Smirnov’s work on the two-dimensional Ising model

Hugo Duminil-Copin, Université de Genève
Recently much of the progress in understanding two-dimensional phenomena resulted from

- Conformal Field Theory (last 25 years)
- Schramm-Loewner Evolution (last 10 years)

There were very fruitful interactions between mathematics and physics. We will try to describe parts of the relations between these three subjects.

**Plan:**

1. Brief historic
2. Study of discrete models
3. Schramm-Loewner Evolution
The Square lattice Ising model

Each cell of a square lattice is either red or blue (corresponding respectively to $+$ or $-$). The probability of a configuration $\sigma$ is

$$P_\beta(\sigma) = \frac{1}{Z_\beta} \exp \left( -\beta \sum_{x \sim y} \sigma(x)\sigma(y) \right)$$

where $Z_\beta$ is the partition function of the model.
Brief historic (1): Onsager’s exact solution

- **1944 (Onsager)**: computation of the partition function (unrigorous)

Other approaches developed:

1. Kac, Ward, Potts, Dolbilin, Cimasoni: combinatorial approach
2. Kasteleyn, Fisher: dimer approach
3. Lieb, Baxter: transfer matrices approach

All these approaches deal with simple geometries (whole plane, torus, etc...). Derive analytic properties. Harder to get geometric ones (in particular dependence on boundary conditions).

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Brief historic (2): Renormalization Group

- 1951, Petermann-Stueckelberg

Perform a block-spin renormalization which corresponds to a rescaling: define an operator from the set of hamiltonians in itself.

Conclusion: At criticality, the scaling limit is described by a mass-less field theory. The critical point is universal and hence translational, scaling and rotational invariant.

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Brief historic (3): Birth of 2D conformal field theory

**Definition:** Conformal transformations are preserving the angles, or in other words are **locally** translations $+$ rotation $+$ scaling.

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⚠️ We must assume the RG is true. It does not address any boundary conditions issue.
Brief historic (4): 2D conformal field theory

- **1966**, *(Patashinkii-Pokrovskii, Kadanoff)* scale, rotation and translation invariance allow to calculate two-point correlations

- **1970**, *(Polyakov)*: Möbius invariance allows to calculate three-point correlations

- **1984**, *(Belavin, Polyakov, Zamolodchikov)* postulate full conformal invariance allows to compute much more things

- **1984**, *(Cardy)* work with boundary fields which leads to applications to lattice models.

- Highest weight of Virasoro's algebra, Quantum gravity, etc...

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Brief historic (6): Geometric and analytic approach

- 1999 (Schramm) Schramm-Loewner Evolution, a geometric description of the scaling limits at criticality
- Recent years (Smirnov) Discrete analyticity: a way to rigorously establish existence and conformal invariance of scaling limits.
DONE

- Brief historic
Brief historic

The Schramm-Loewner Evolution
Discrete observables and lattice models
What is next
Schramm-Loewner Evolution (pre-history)

Event 1 (1994): *(Langlands, Pouilot, Saint-Aubin)* check the existence of the limit, the universality and the conformal invariance of crossing probabilities for percolation.

Very widely read

Event 2: *(Cardy)* crossing formula for percolation:

\[
\lim_{\delta \to 0} P_\delta (C(R)) = \frac{\Gamma \left( \frac{2}{3} \right)}{\Gamma \left( \frac{4}{3} \right) \Gamma \left( \frac{1}{3} \right)} m^{\frac{1}{3}} \ 2F_1 \left( \frac{1}{3}, \frac{2}{3}, \frac{4}{3}, m \right),
\]

where \( m \) is the conformal radius of the rectangle.

Easier version by Carleson, proved by Smirnov (2001)
(Schramm) Look at interfaces in models of statistical physics.
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Model these interfaces at the scaling limit by a random continuous curve, called SLE.
Schramm-Loewner Evolution (pre-history)

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Model these interfaces at the scaling limit by a random continuous curve, called SLE.

It allows him to deduce (among other things) Cardy’s formula, conditionally to the fact that the discrete interface indeed converges to SLE.
Schramm-Loewner Evolution (definition Loewner chain)

Consider a simply connected domain $D$ with two points on the boundary (for instance think of $(\mathbb{H}, 0, \infty)$) and a growing curve from 0 to $\infty$.

$(Loewner 1920)$ Growing curves can be coded by real functions.
**Schramm-Loewner Evolution (definition Loewner chain)**

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  \[(\text{Loewner 1920})\] Growing curves can be coded by real functions.

\[ g : H \setminus \gamma \to H \quad \infty \]
\[ z \mapsto z + \frac{C}{z} + O(1/z^2) \]

Moreover for every $z \in H$, up to the first time at which $z$ is swallowed by the curve, we have:

\[ \partial_t g_t(z) = 2g_t(z) - W_t. \]

The process $W_t$ is called the driving process.
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\[ g_t : H \setminus \gamma[0, t] \rightarrow H \]
\[ z \mapsto z + \frac{2t}{z} + O(1/z^2) \]

Moreover for every $z \in H$, up to the first time at which $z$ is swallowed by the curve, we have:

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Moreover for every $z \in \mathbb{H}$, up to the first time at which $z$ is swallowed by the curve, we have:

$$\partial_t g_t(z) = \frac{2}{g_t(z) - W_t}.$$

The process $W_t$ is called the driving process.
Observation 2: (domain Markov property) Conditionally on the start of the curve, the remaining curve is an SLE from the tip to $\infty$ in the slit domain.
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Observation 2: curves must be conformally invariant:

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\Phi(a) \rightarrow \Phi(b) \quad \Phi(D) \rightarrow \Phi(\gamma) \quad \Phi(\gamma) \rightarrow \Phi'(\gamma)
\]

With these two properties, the driving process must have independent increments. Scale invariance implies that it is a Brownian motion.

The SLE($\kappa$) is the (random) Loewner chain generated by the $\sqrt{\kappa}B_t$ ($B_t$ is a standard Brownian motion).

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Schramm-Loewner Evolution (properties of SLE itself)

A fractal curve:
- Simple for \( \kappa \leq 4 \)
- Self-touching for \( \kappa \in (4, 8) \)
- Space filling for \( \kappa \geq 8 \) (Rohde, Schramm)

Hausdorff dimension \( \dim_{Hausdorff}(\text{SLE}(\kappa)) = (1 + \kappa) / 8 \) (Beffara)

Computation of critical exponents (like intersection exponents).

Duality properties between \( \kappa \) and \( 16/\kappa \) (Zhan, Dubédat).

Allows to construct more general processes, such as CLEs (Schramm, Sheffield, Werner).

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Schramm-Loewner Evolution (connection with CFT)

$\kappa = 2$
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$\kappa = 8$

$\kappa = 8/3$

loop erased Ising
Dimers
Level lines of GFF
Percolation
UST
SAW

Construction of a highest weight representation of Virasoro’s algebra
(Friedrich, Werner)

Link between SLE martingales and CFTs
(Bauer, Bernard)

Gaussian free field

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  \[ \kappa = \frac{8}{3}, \kappa = 2, \kappa = 3, \kappa = 4, \kappa = 6, \kappa = 8 \]

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Smirnov’s work on the two-dimensional Ising model
Brief historic

The Schramm-Loewner Evolution
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The Schramm-Loewner Evolution

TO DO

Discrete observables and lattice models

What is next
Discrete observables (General philosophy)

Consider a family of interfaces between two points $a$ and $b$ in (discrete approximations with meshsize $\delta$ of) a fixed domain $\Omega$.
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To identify the possible curve, one needs explicit martingales...
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To identify the possible curve, one needs explicit martingales...

(1) These martingales should be **observables** of the lattice model (crossing probabilities, magnetization, etc)...
(2) We can compute them because they are discrete holomorphic and they satisfy some fixed boundary problem.
(3) Hence, they converge to the continuum analogue of the problem.

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Smirnov’s work on the two-dimensional Ising model
Discrete observables (The case of the Ising model on the triangular lattice)

Consider the hexagonal lattice for a second and define the high-temperature expansion of the Ising model.
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For a simply-connected domain $\Omega$ and a discrete approximation of it, let $z$ be on the boundary. Define the fermionic operator

$$F_{\Omega, \delta, x}(a, z) = \sum_{\omega \text{ with a curve } \gamma \text{ from } a \text{ to } z} e^{-i \frac{1}{2} W_{\gamma}(a, z)} x^{\#\text{edges}}.$$
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If $x = x_c$, the function $z \mapsto F_{\Omega,\delta,x_c}(a, z)$ is a discrete Green function with Riemann-Hilbert boundary-value problem.
Discrete observables (The case of the Ising model on the triangular lattice)

Let $\Omega$ be a simply connected domain and $a$, $b$ on the boundary.

**Fact 1:**
For $z$ inside the domain,\[
\lim_{\delta \to 0} F_{\Omega, \delta, x_c}(a, z) = \sqrt{K'_{\Omega}(a, z)} \cdot K'_{\Omega}(a, b).
\]

**Fact 2:**
The quantity $F_{\Omega, \gamma_t, \delta, x_c}(\gamma_t, z) = F_{\Omega, \gamma_t, \delta, x_c}(\gamma_t, b)$ is a martingale (conserved quantity) of the discrete curve from $a$ to $b$.

When plugging that for each $z$, $\sqrt{K'_{\Omega}(a, z)} \cdot K'_{\Omega}(a, b)$ is a conserved quantity of the limiting curve, we deduce that the only possible limit is SLE(3)!
Discrete observables (The case of the Ising model on the triangular lattice)

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**Fact 2:** The quantity

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\frac{F_{\Omega \setminus \gamma_t, \delta, x_c}(\gamma_t, z)}{F_{\Omega \setminus \gamma_t, \delta, x_c}(\gamma_t, b)}
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is a martingale (conserved quantity) of the discrete curve from $a$ to $b$. 

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Discrete observables (Implications)

convergence to SLE(3) (Chelkak, Smirnov) which leads to exponents understanding of the geometric properties (D.-C., Hongler and Nolin), leading to mixing estimates for the Glauber dynamics (Sly, Lubetzky)

construction of the energy density field (Hongler, Smirnov):

$$\langle \sigma(x) \sigma(y) \rangle_{\Omega, \epsilon, \text{free}} = \sqrt{2} - \frac{1}{\pi \rho} \Omega(x) \epsilon + O(\epsilon^2).$$

Smirnov's work on the two-dimensional Ising model
Discrete observables (Implications)

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- understanding of the geometric properties (D.-C., Hongler and Nolin), leading to mixing estimates for the Glauber dynamics (Sly, Lubetzky)

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\langle \sigma(x) \sigma(y) \rangle_{\Omega, \varepsilon, \text{free}} = \sqrt{2} - \frac{1}{\pi \rho_{\Omega}(x)} + O(\varepsilon^2).
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Discrete observables (Other $O(n)$-models)

The $O(n)$ model is a model on **closed loops** lying on a finite subgraph of the hexagonal lattice: the partition function equals

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where $2 \cos\left(\frac{4(1/2+\sigma)\pi}{3}\right) = -n$. 

Hugo Duminil-Copin, Université de Genève

Smirnov’s work on the two-dimensional Ising model
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Smirnov’s work on the two-dimensional Ising model
Brief historic
The Schramm-Loewner Evolution
Discrete observables and lattice models
DONE

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TO DO

- What is next
Prove conformal invariance of other lattice models
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- Comprehend links between SLE (or CLE) and CFT deeper
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Relate random planar graphs to Liouville Quantum Gravity via SLE.
Thank you