Smirnov's work on the two-dimensional Ising model

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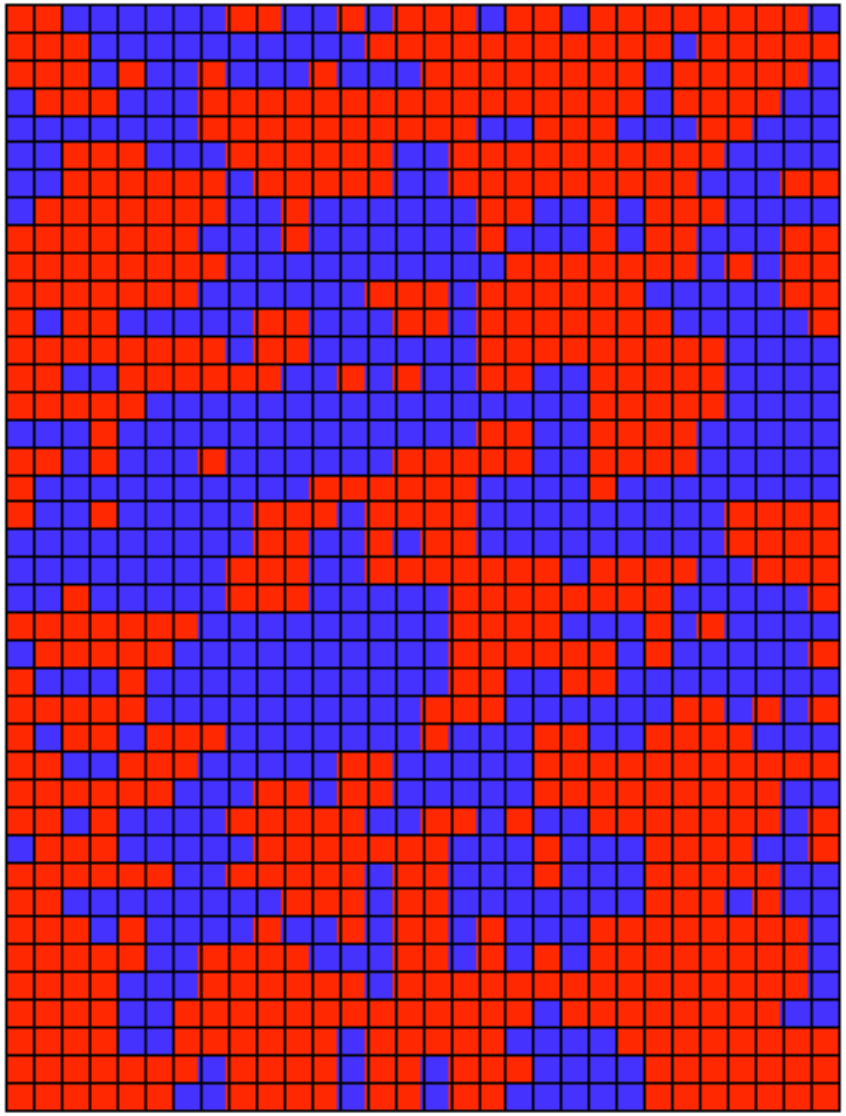
Recently much of the progress in understanding two-dimensional phenomena resulted from

- Conformal Field Theory (last 25 years)
- Schramm-Loewner Evolution (last 10 years)

There were very fruitful interactions between mathematics and physics. We will try to describe parts of the relations between these three subjects.

Plan:
(1) Brief historic
(2) Study of discrete models
(3) Schramm-Loewner Evolution

The Square lattice Ising model



Each cell of a square lattice is either red or blue (corresponding respectively to $+$ or $-$). The probability of a configuration σ is

$$\mathbb{P}_\beta(\sigma) = \frac{1}{Z_\beta} \exp\left(-\beta \sum_{x \sim y} \sigma(x)\sigma(y)\right)$$

where Z_β is the partition function of the model.

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⚠ All these approaches deal with simple geometries (whole plane, torus, etc...). Derive analytic properties. Harder to get geometric ones (in particular dependence into boundary conditions).

Brief historic (2): Renormalization Group

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We must believe in this picture (for instance for the problem of marginal 'operators').

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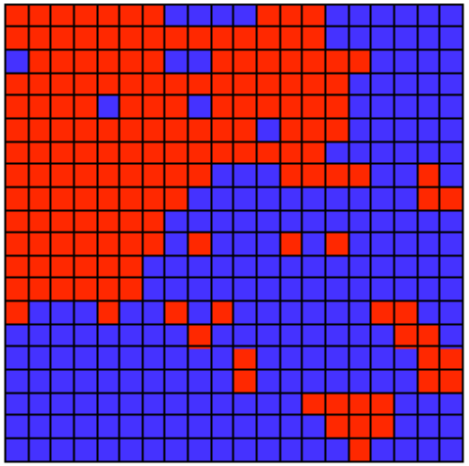
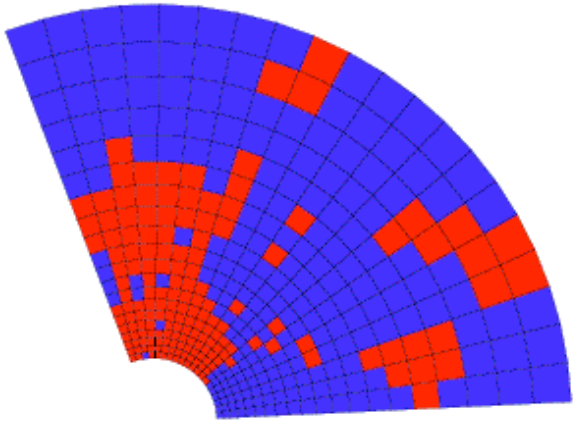
Definition: Conformal transformations are preserving the angles, or in other words are **locally translations+rotation+scaling**.

Since the fields are local, one can logically postulate the conformal invariance of the model.





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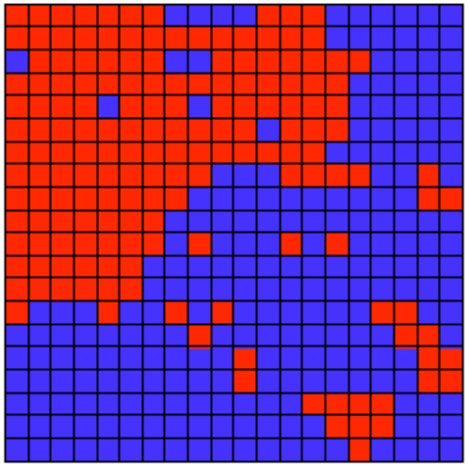

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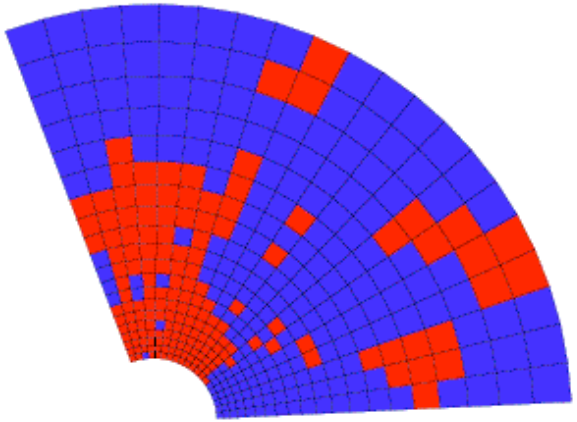

We must assume the RG is true. It does not adress any boundary conditions issue.

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- highest weight of Virasoro's algebra, Quantum gravity, etc...

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Brief historic (6): Geometric and analytic approach

- **1999** *(Schramm)* Schramm-Loewner Evolution, a geometric description of the scaling limits at criticality
- **Recent years** *(Smirnov)* Discrete analyticity: a way to rigorously establish existence and conformal invariance of scaling limits.

DONE

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TO DO

- The Schramm-Loewner Evolution
- Discrete observables and lattice models
- What is next

Schramm-Loewner Evolution (pre-history)

Event 1 (1994): *(Langlands, Pouilot, Saint-Aubin)* check the existence of the limit, the universality and the conformal invariance of crossing probabilities for percolation.

Very widely read

Event 2: *(Cardy)* crossing formula for percolation:

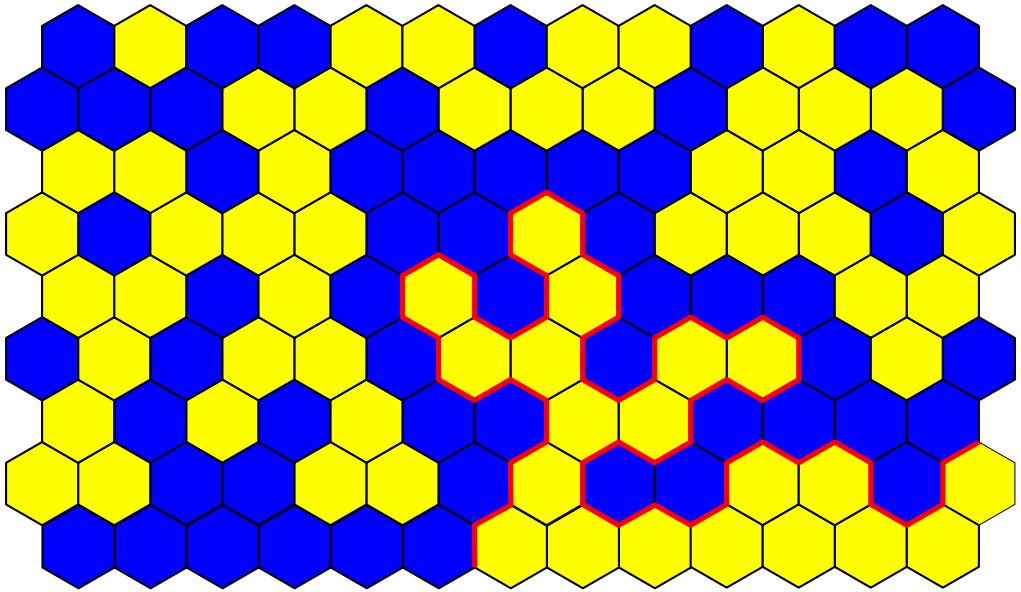
$$\lim_{\delta \to 0} P_\delta\big(\mathcal{C}(R)\big) = \frac{\Gamma\left(\frac{2}{3}\right)}{\Gamma\left(\frac{4}{3}\right)\Gamma\left(\frac{1}{3}\right)} m^{\frac{1}{3}} \; {}_2F_1\left(\frac{1}{3}, \frac{2}{3}, \frac{4}{3}, m\right),$$

where m is the conformal radius of the rectangle.

Easier version by Carleson, proved by Smirnov (2001)

Schramm-Loewner Evolution (pre-history)

(Schramm) Look at interfaces in models of statistical physics.



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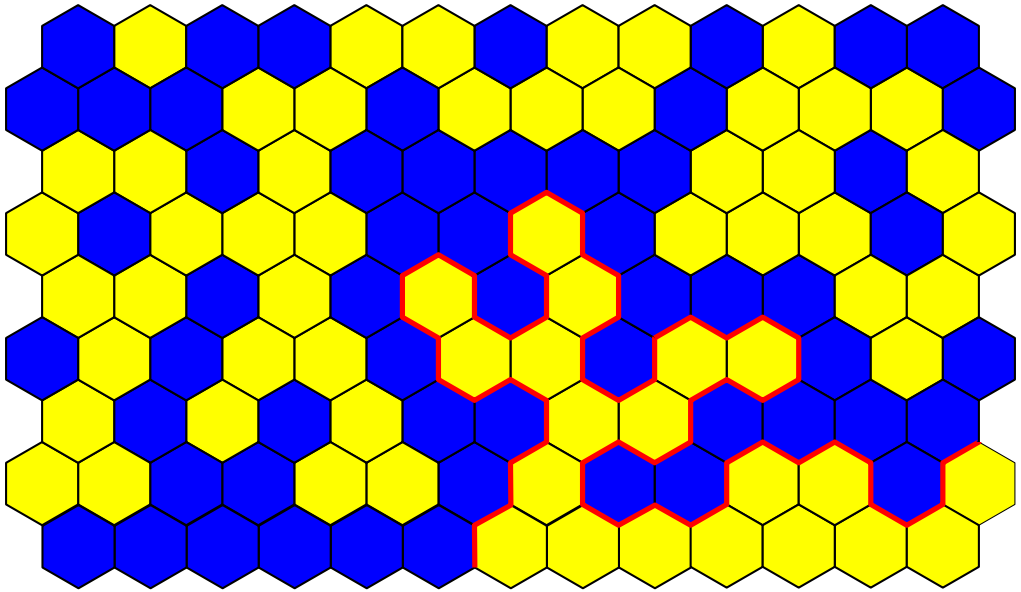

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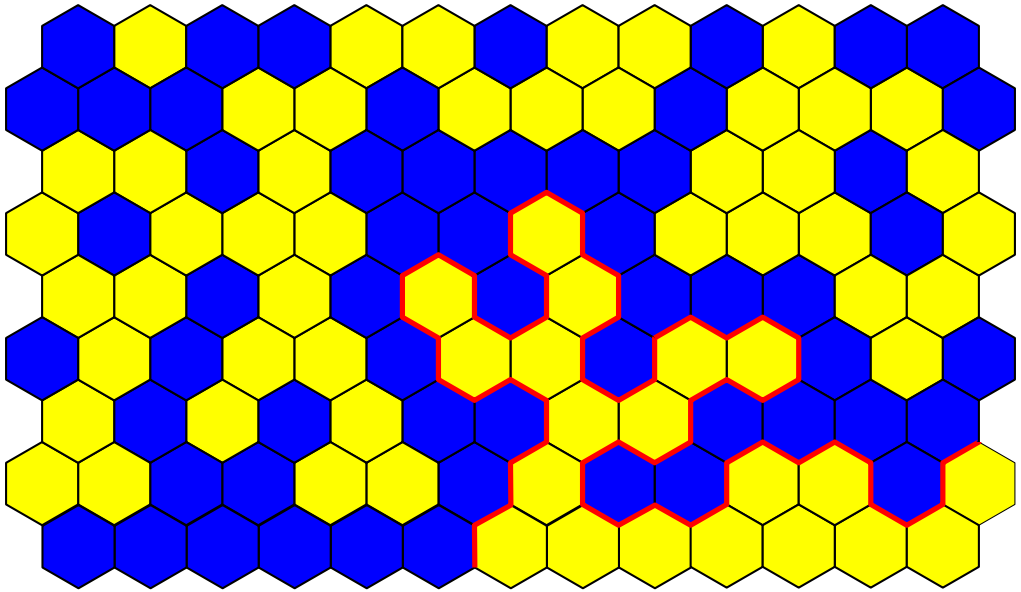

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It allows him to deduce (among other things) Cardy's formula, **conditionally to** the fact that the discrete interface indeed converges to SLE.

Schramm-Loewner Evolution (definition Loewner chain)

- Consider a simply connected domain $\mathcal{D}$ with two points on the boundary (for instance think of $(\mathbb{H}, 0, \infty)$) and a growing curve from 0 to ∞.

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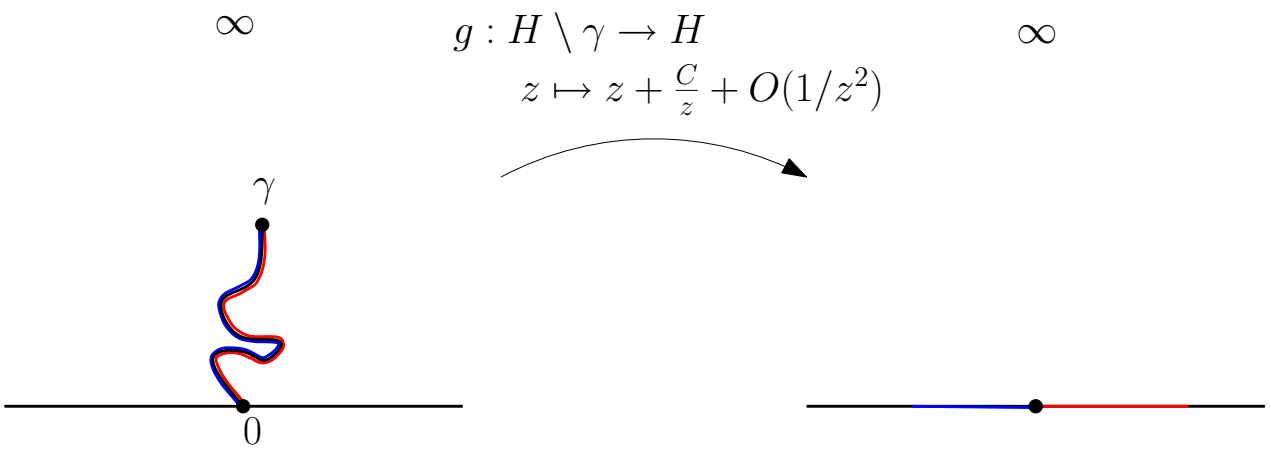

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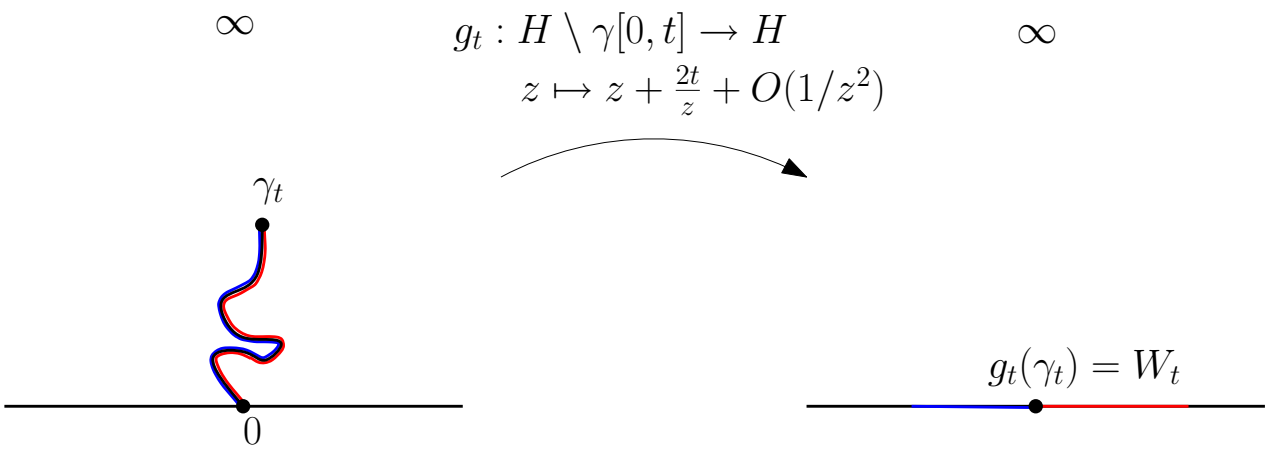

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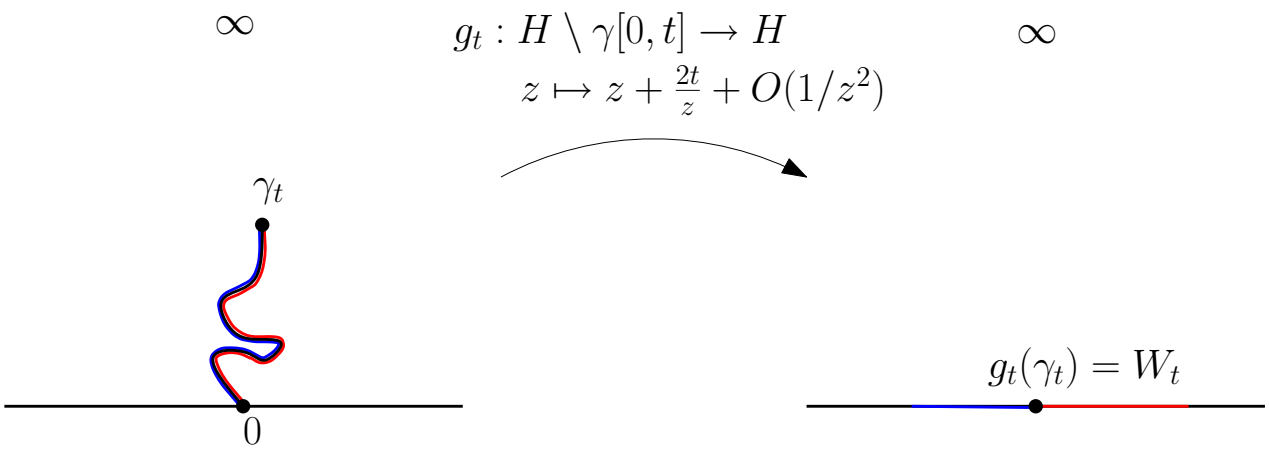

- Moreover for every $z \in \mathbb{H}$, up to the first time at which z is swallowed by the curve, we have:

$$\partial_t g_t(z) = \frac{2}{g_t(z) - W_t}.$$

The process W_t is called the driving process.

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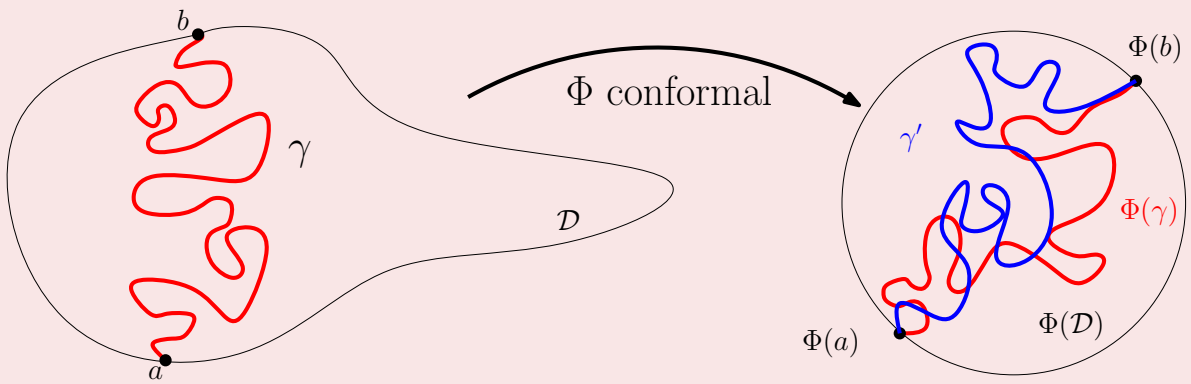

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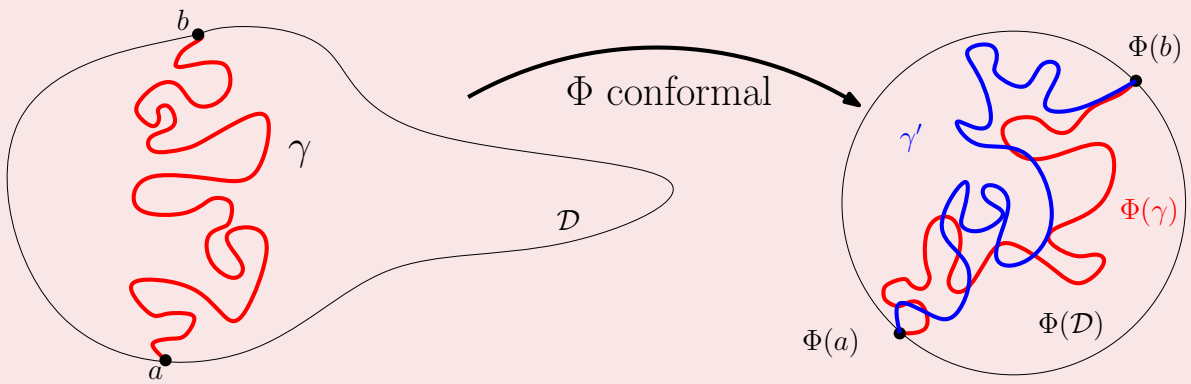

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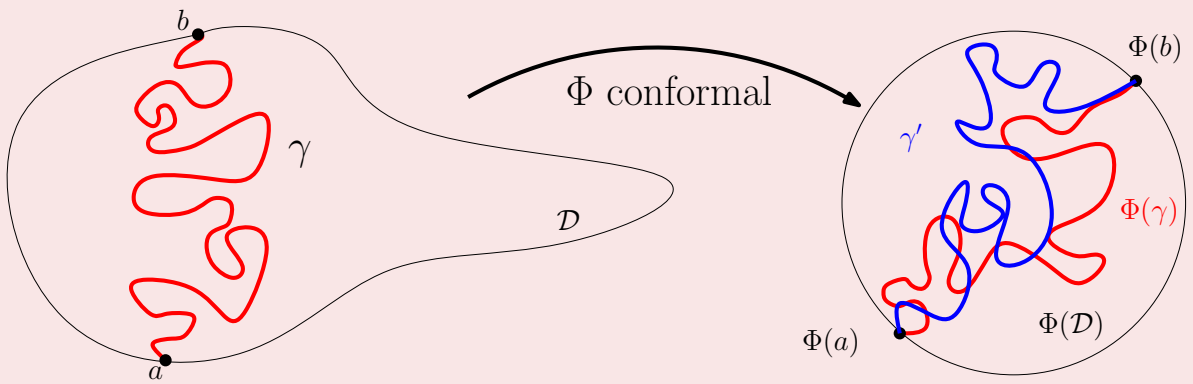

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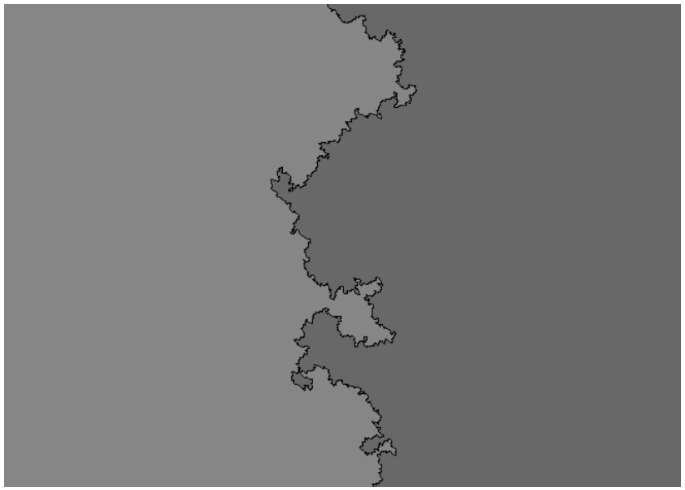
Schramm-Loewner Evolution (properties of SLE itself)



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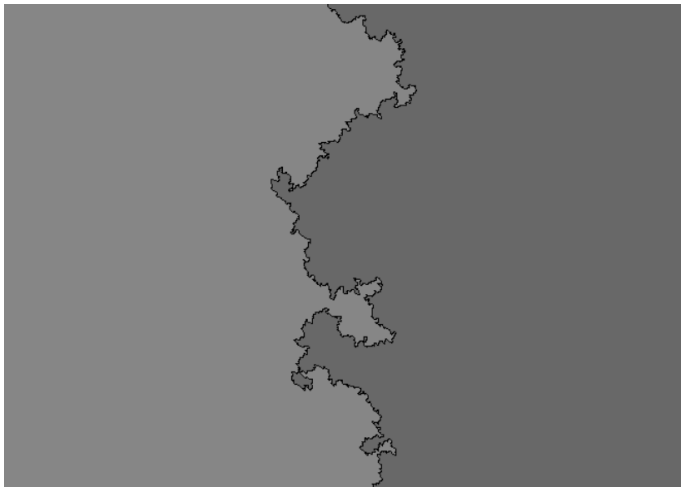

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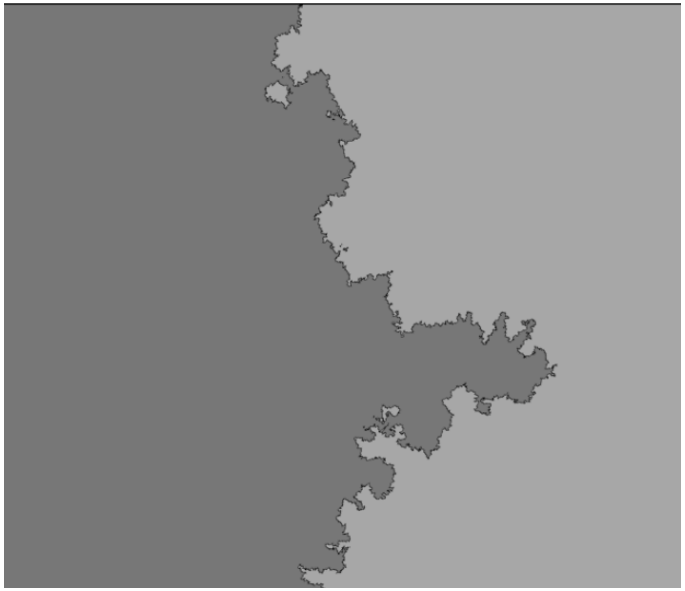

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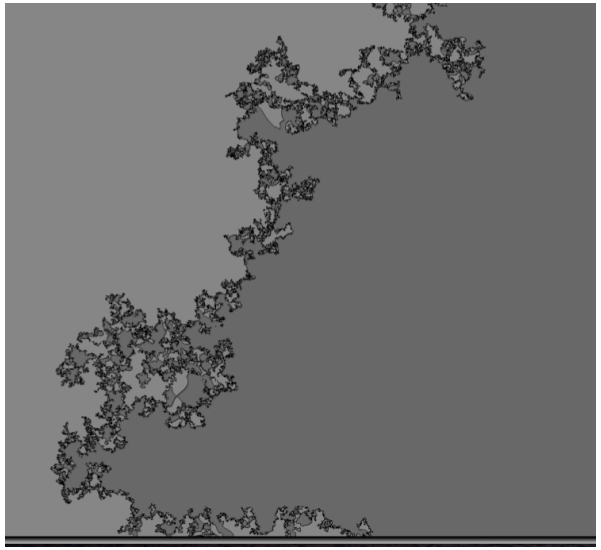

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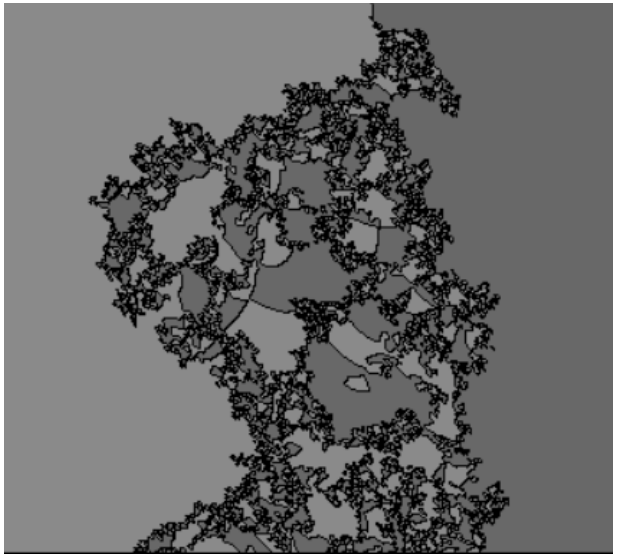

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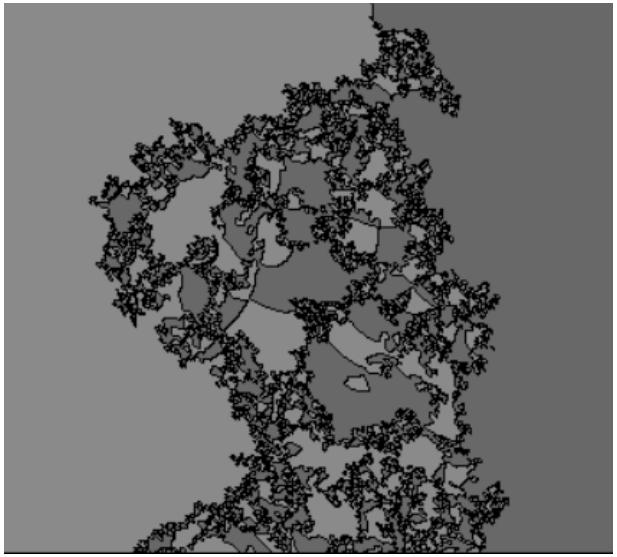

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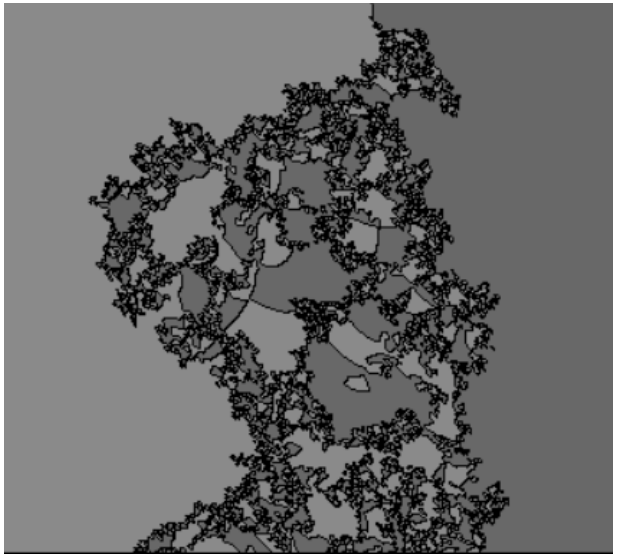

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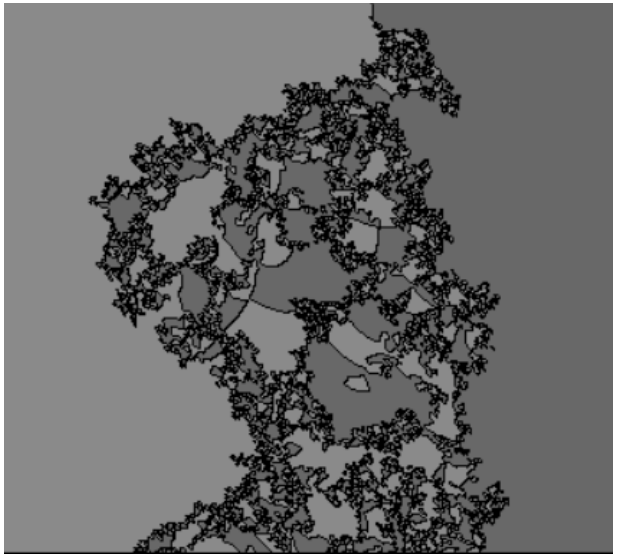

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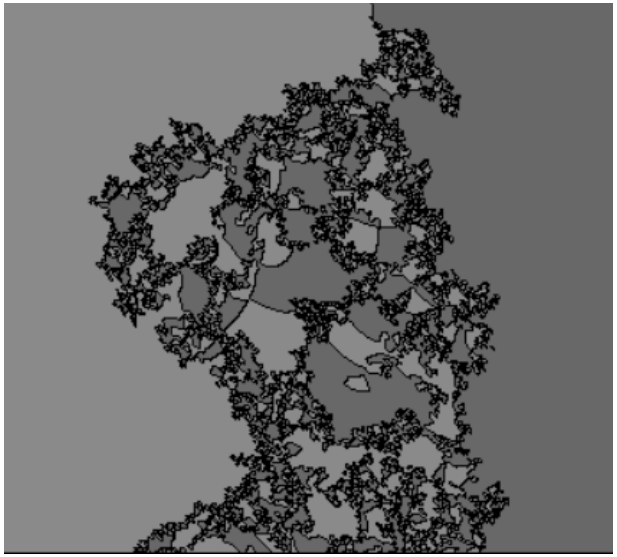

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- allows to construct more general processes, such as CLEs *(Schramm, Sheffield, Werner)*.

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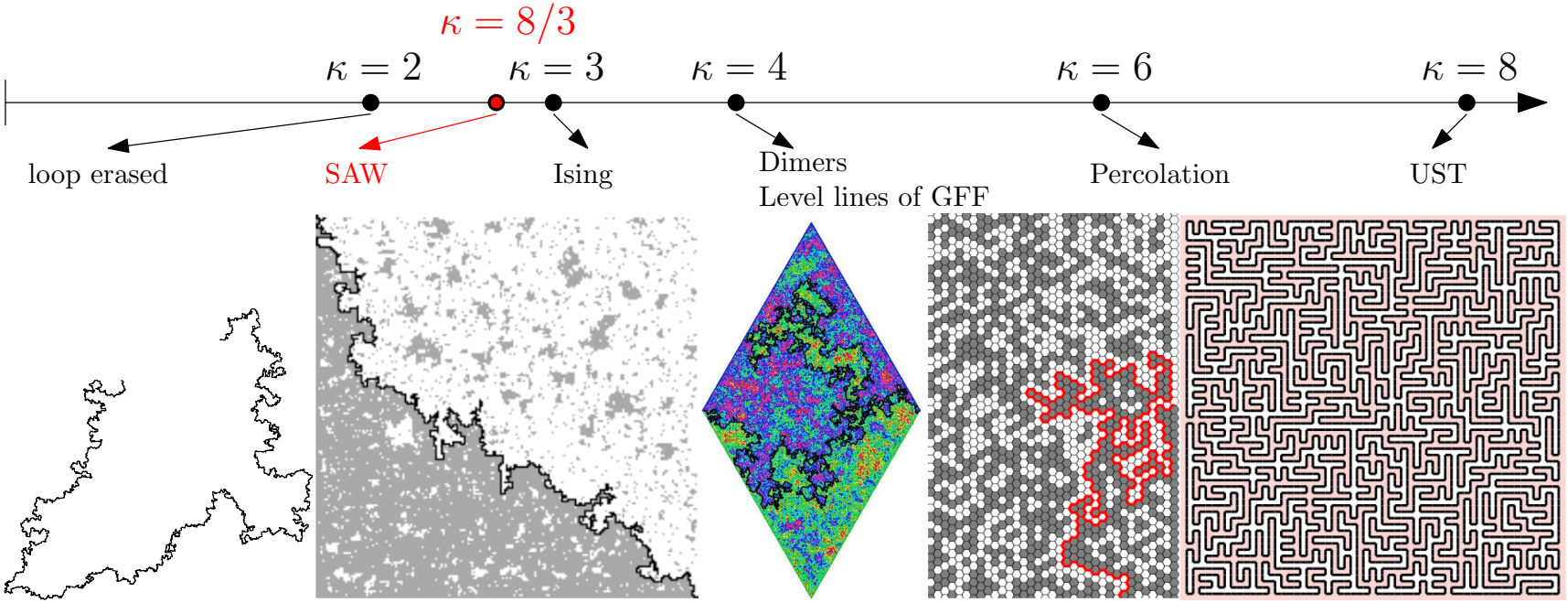

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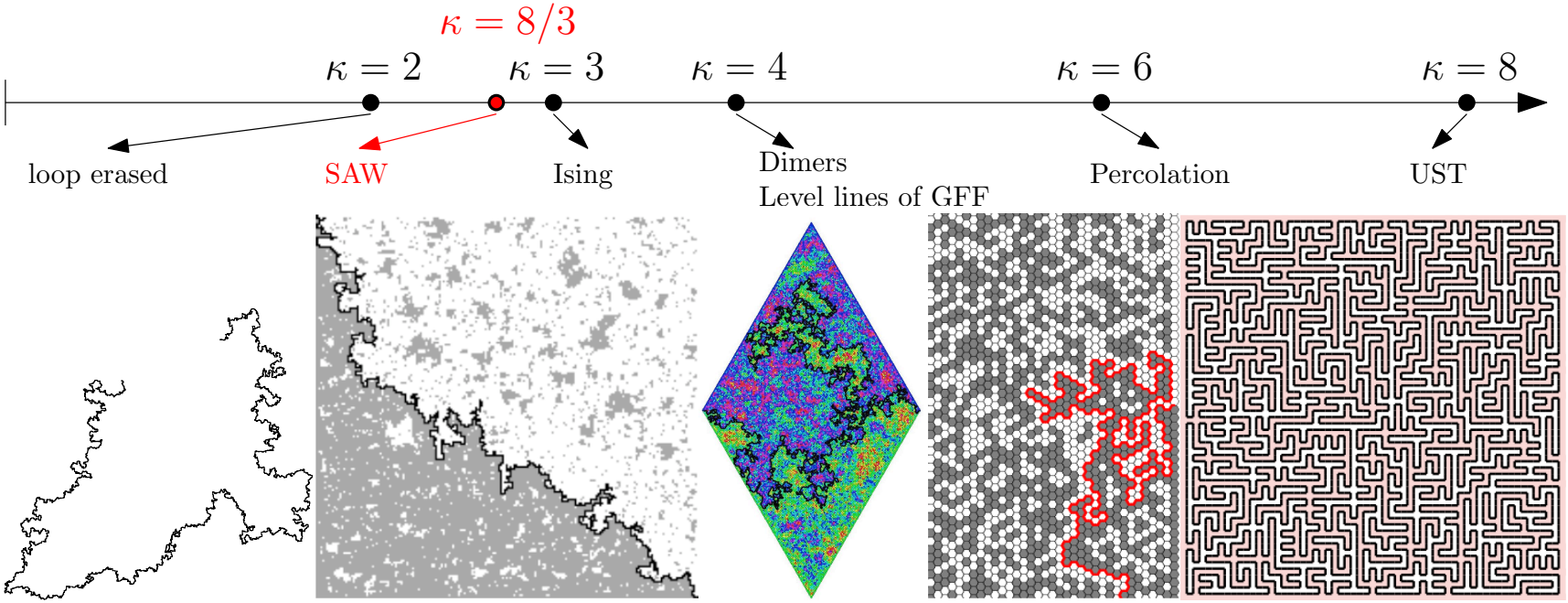

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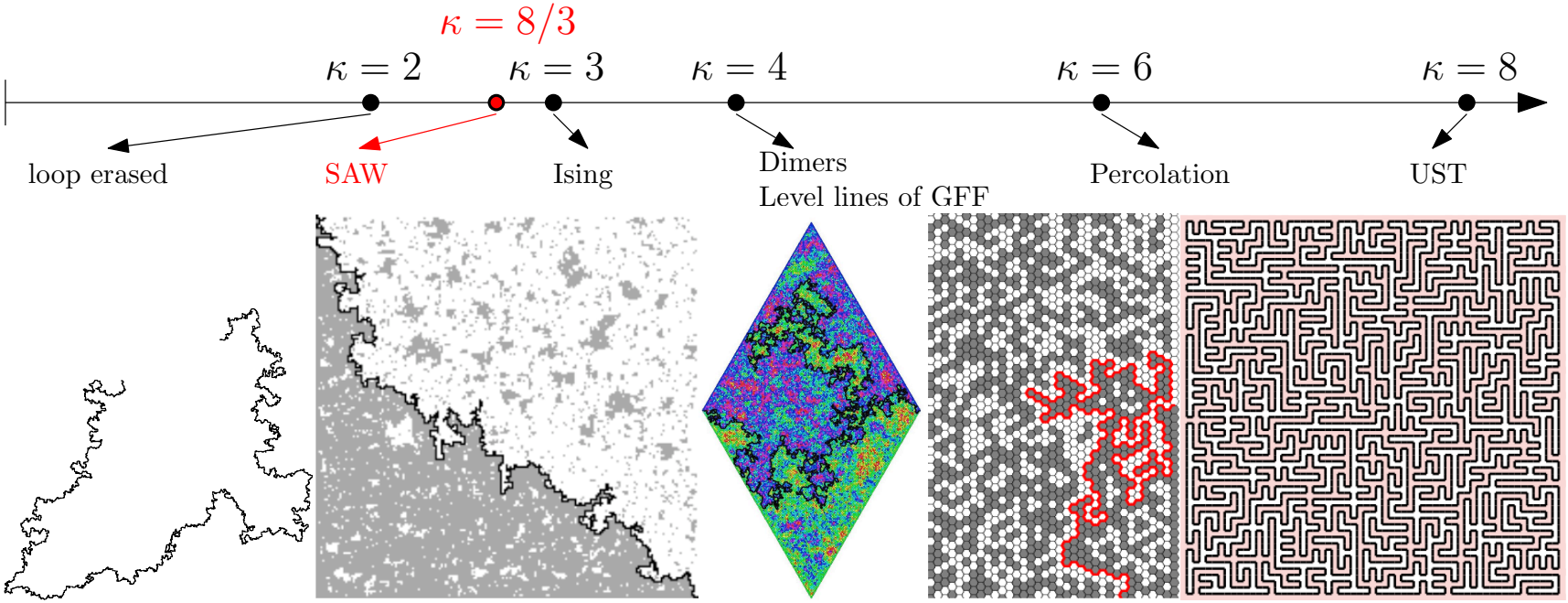

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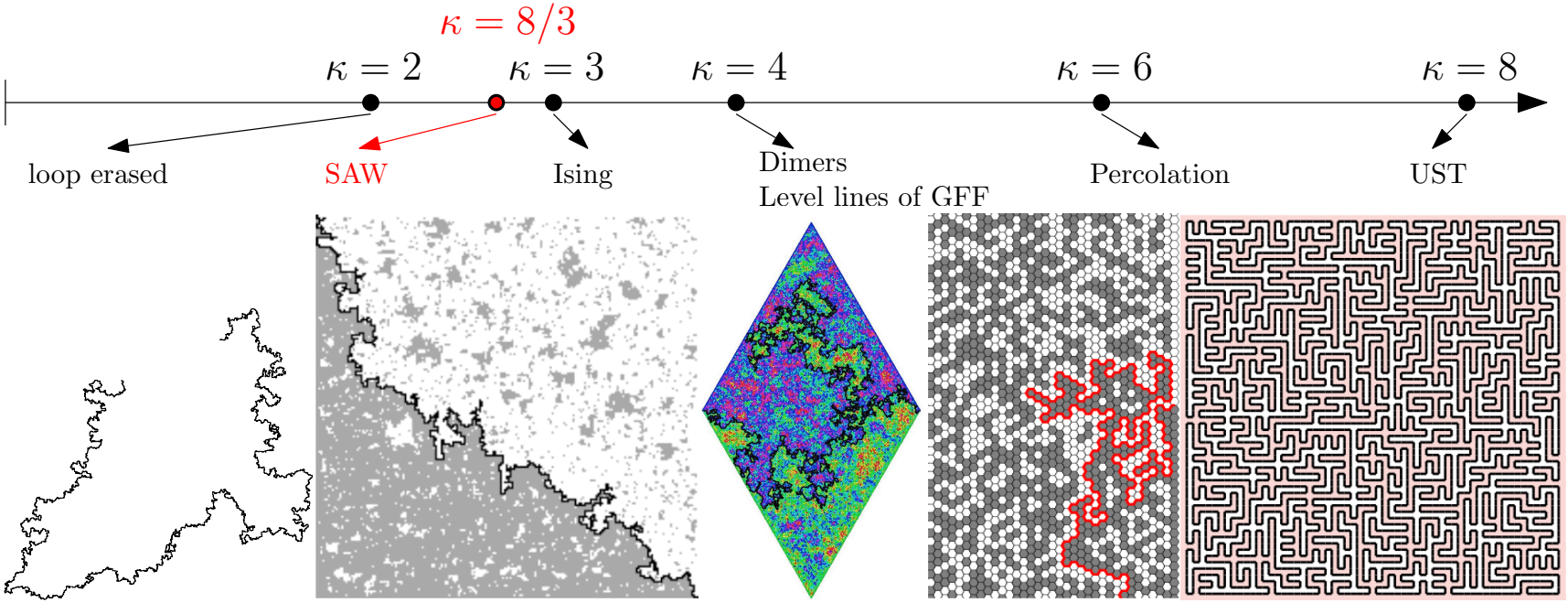

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- Gaussian free field

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- Discrete observables and lattice models
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To identify the possible curve, one needs explicit martingales...

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- Consider a family of interfaces between two points a and b in (discrete approximations with meshsize δ of) a fixed domain Ω.
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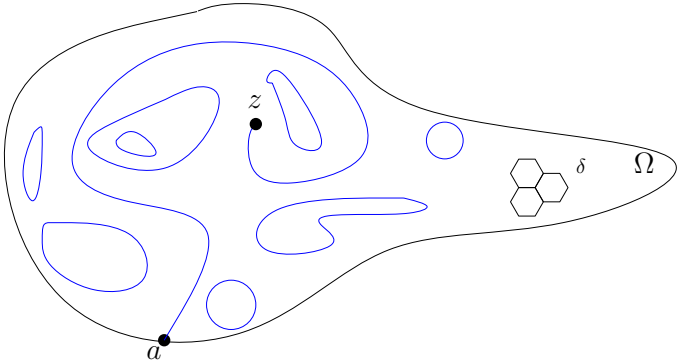
To identify the possible curve, one needs explicit martingales...

(1) These martingales should be **observables** of the lattice model (crossing probabilities, magnetization, etc)...
(2) We can compute them because they are discrete holomorphic and they satisfy some fixed boundary problem.
(3) Hence, they converge to the continuum analogue of the problem.

Discrete observables (The case of the Ising model on the triangular lattice)

Consider the hexagonal lattice for a second and define the high-temperature expansion of the Ising model.



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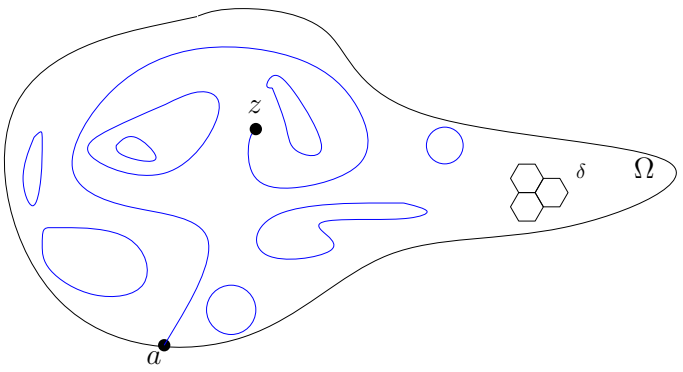

For a simply-connected domain Ω and a discrete approximation of it, let z be on the boundary. Define the fermionic operator

$$F_{\Omega,\delta,x}(a,z) = \sum_{\omega \text{ with a curve } \gamma \text{ from } a \text{ to } z} e^{-i\frac{1}{2}W_\gamma(a,z)} x^{\#\text{edges}}.$$

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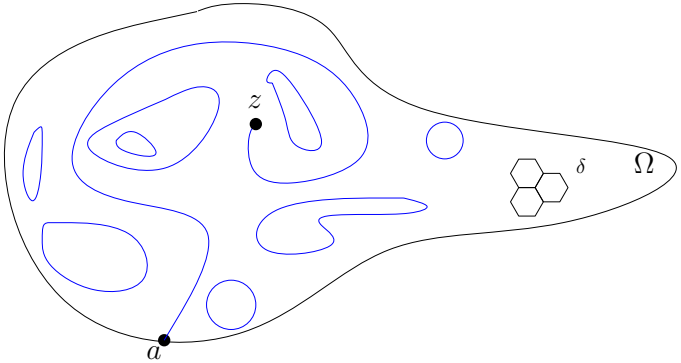

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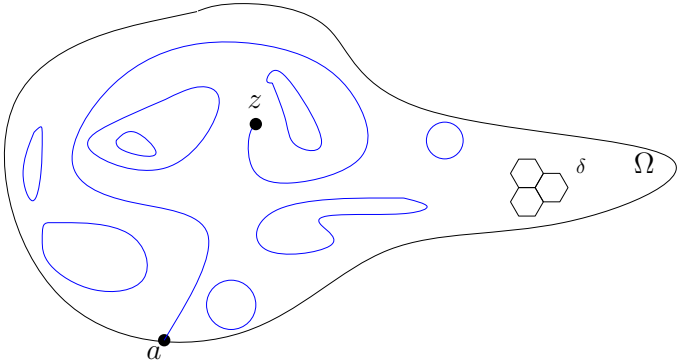

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If $x = x_c$, the function $z \mapsto F_{\Omega,\delta,x_c}(a,z)$ is a discrete Green function with Riemann-Hilbert boundary-value problem.

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When plugging that for each z, $\sqrt{\frac{K'_{\Omega\setminus\gamma_t}(a,z)}{K'_{\Omega\setminus\gamma_t}(a,b)}}$ is a conserved quantity of the limiting curve, we deduce that the only possible limit is SLE(3)!

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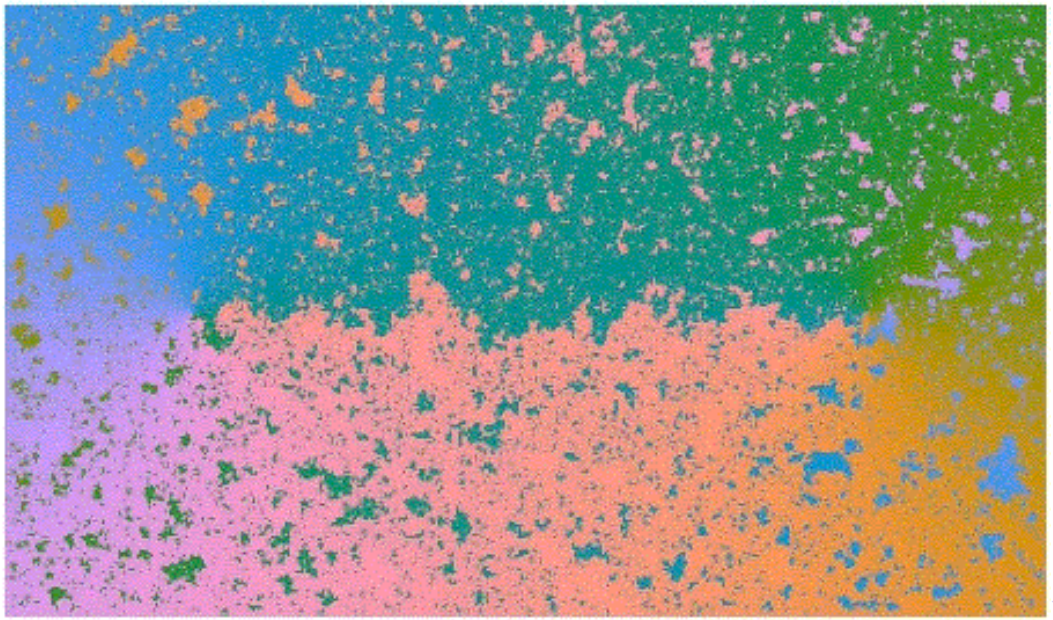
- convergence to SLE(3) (Chelkak, Smirnov) which leads to exponents
- understanding of the geometric properties (D.-C., Hongler and Nolin), leading to mixing estimates for the Glauber dynamics (Sly, Lubetzky)
- construction of the energy density field (Hongler, Smirnov):

$$\langle \sigma(x)\sigma(y)\rangle_{\Omega_\varepsilon,free} = \frac{\sqrt{2}}{2} - \frac{1}{\pi}\rho_\Omega(x)\varepsilon + O(\varepsilon^2).$$



Discrete observables (Other $O(n)$-models)

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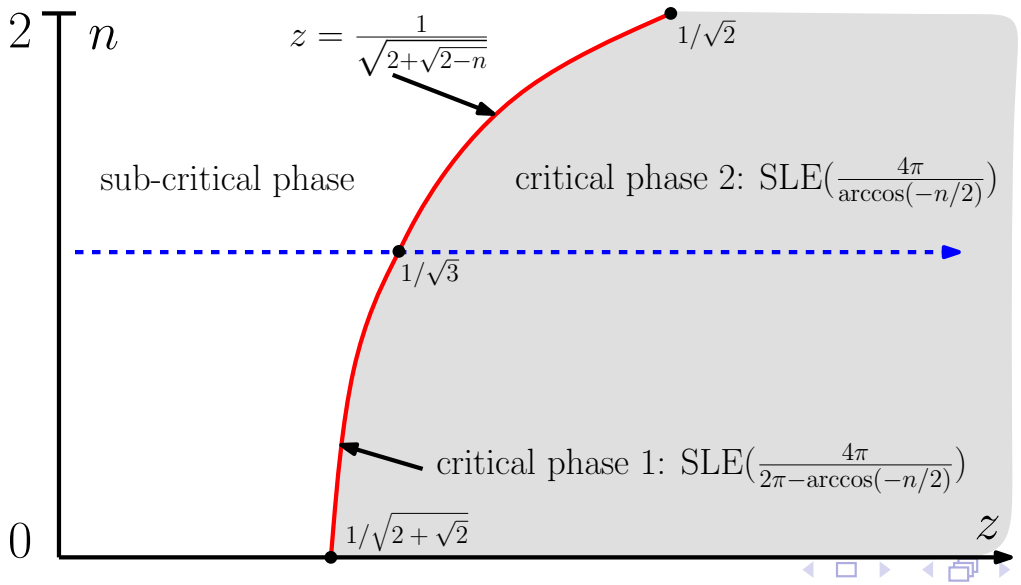

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TO DO

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Thank you