Random Matrices and topological strings

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GENEZISS String Theory Meeting, EPFL, Lausane

based on collaborations with A. Kashani-Poor, O. Marchal, and L. Chekhov, N. Orantin, M. Mariño, A. Pratts-Ferrer.

november 26th 2010

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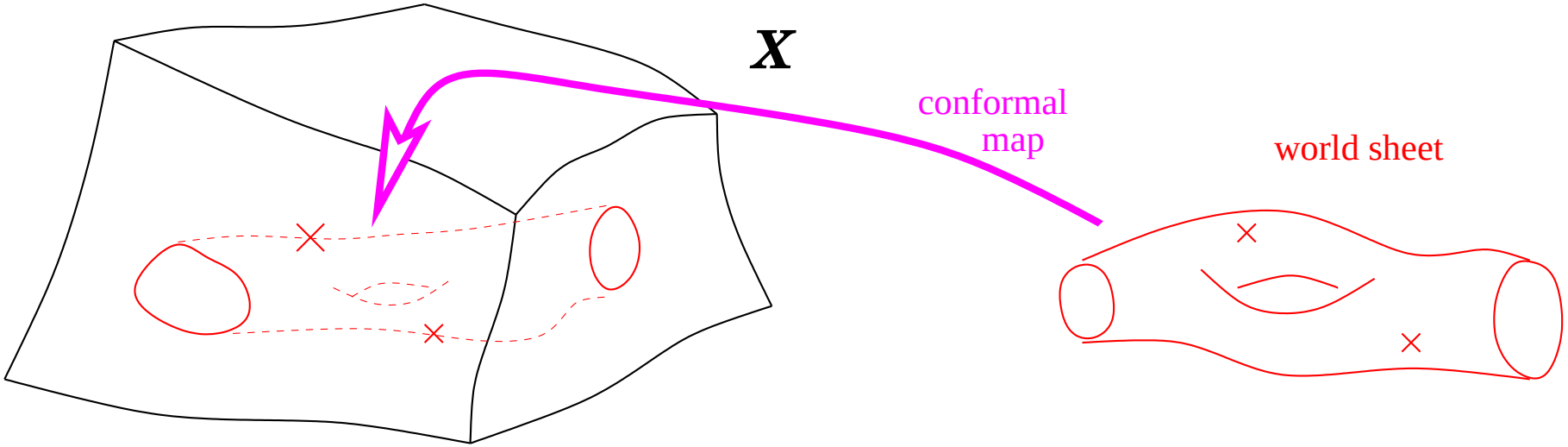
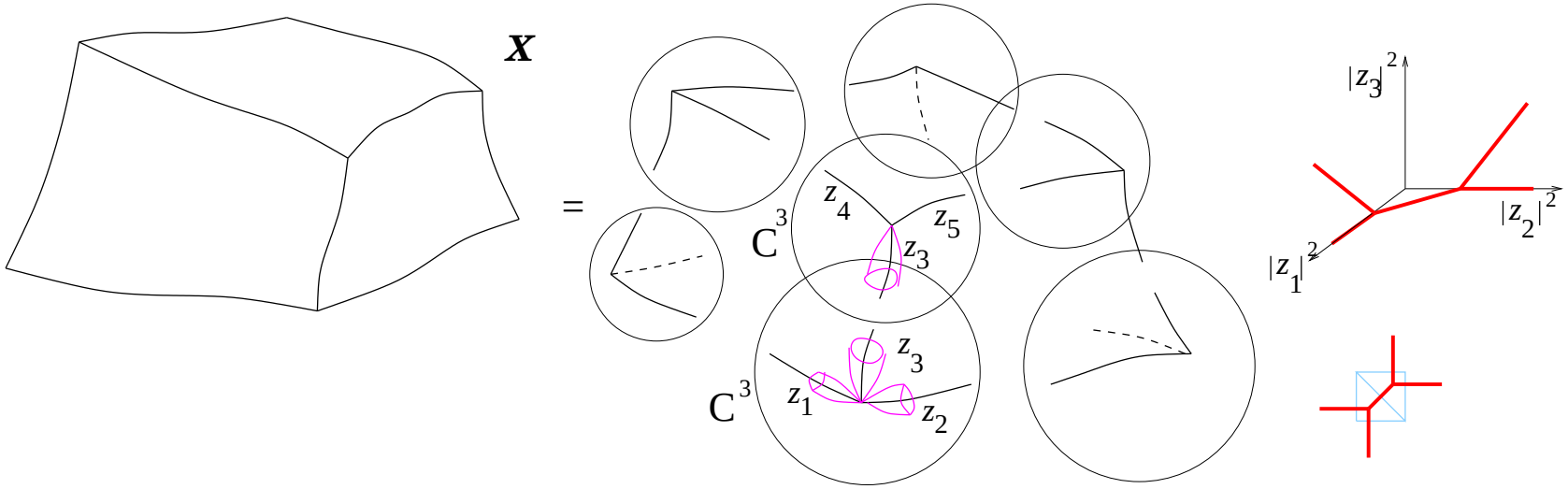
Topological strings



- Type IIA, topological string theory in a target space $\mathfrak{X}$, which we choose to be a CY 3-fold, with Toric symmetry.
- Useful for: some low energy limit of string theory, captures some exact features of string theory. Applications to mathematics: algebraic geometry, Gromov-Witten invariants. Application to knot invariants, equivalence with Chern-Simons theory.

Toric symmetry and topological vertex

CY3-fold can be obtained by gluing $\mathbb{C}^3$ patches.



$\mathbb{C}^3$ has a cannonical T^3 torus action $z_i \to z_i e^{i\theta_i}$, which degenerates when $|z_i|^2 = 0$. The lines where two toric symmetries degenerate draw a diagram in $\mathbb{R}^3_+$, project it to $\mathbb{R}^2$ = Toric diagram.

String amplitudes are computed by "gluing" $\mathbb{C}^3$ open string amplitudes, and summing over gluing moduli: $\to$ topological vertex formula [AKMV 2003].

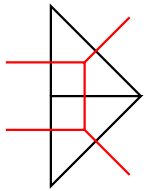
Toric symmetry and topological vertex

$\Rightarrow$ All A-model topological string theory partition functions in toric CY3 target spaces, can be written as sums over partitions, in a way encoded by the toric diagram.

Goal: • rewrite it as a matrix integral.

• see how mirror symmetry emerges from it, remodelling the B-model, topological recursion, integrability, holomorphic annomaly, non-perturbative part, ...

Topological vertex, example 1



Example 1: Nekrasov's SW $SU(1)$ partition function

$$Z_{4d} = \sum_{\lambda} \left(\frac{\dim \lambda}{|\lambda|!} \right)^2 \mathrm{e}^{-t\,|\lambda|}$$

note $h_i = \lambda_i - i + N$

$$= \sum_{h_1 > \cdots > h_N \geq 0} \frac{\Delta(h)^2}{\prod_i h_i!^2} \prod_i \mathrm{e}^{-t \sum_i h_i}$$

$$\Delta(h) = \prod_{i>j} (h_i - h_j)$$

$$= \frac{1}{N!} \sum_{h_1, \ldots, h_N \geq 0} \frac{\Delta(h)^2}{\prod_i h_i!^2} \prod_i \mathrm{e}^{-t \sum_i h_i}$$

$$= \frac{1}{N!} \oint_{\mathcal{C}^N} dh_1 \ldots dh_N \; \Delta(h)^2 \prod_i \frac{\Gamma(-h_i)\, \mathrm{e}^{i\pi h_i}}{\Gamma(h_i + 1)} \mathrm{e}^{-t \sum_i h_i}$$

$$= \frac{1}{N!} \int_{H_N(\mathcal{C})} dH \; \frac{\det\left(\Gamma(-H)\right) \mathrm{e}^{\mathrm{i}\pi \operatorname{Tr} H}}{\det\left(\Gamma(1 + H)\right)} \mathrm{e}^{-t \operatorname{Tr} H}$$

$$H = UhU^\dagger$$

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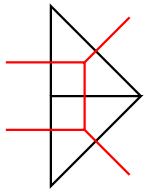

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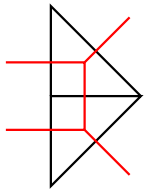

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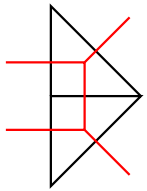

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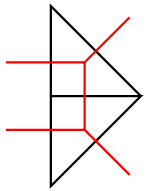

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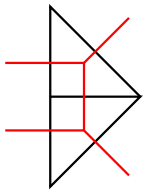
Topological vertex, example 2

in 5-dimension, with radius $q = \mathrm{e}^{-g_s}$, and $h_i = q^{\lambda_i - i + N}$:



$$
\begin{aligned}
Z_{5d} &= \sum_{\lambda} \left(\frac{\dim_q \lambda}{[|\lambda|]_q!} \right)^2 \mathrm{e}^{-t\,|\lambda|} \\
&= \frac{1}{N!} \sum_{h_1,\dots,h_N} \frac{\prod_{i<j}(h_i - h_j)^2}{\prod_i h_i^{N-\frac{1}{2}}\, \Gamma_q(q\, h_i)^2}\, \mathrm{e}^{\frac{-1}{2g_s}\sum_i (\ln h_i)^2}\, \mathrm{e}^{\frac{t}{g_s}\sum_i \ln h_i} \\
&= \frac{1}{N!} \int_{H_N(\mathcal{C})} dH\, \frac{\det(\Gamma_q(1/H))}{\det(\Gamma_q(qH))}\, \frac{\mathrm{e}^{\frac{\mathrm{i}\pi}{g_s}\,\mathrm{Tr}\,\ln H}}{(\det H)^N}\, \mathrm{e}^{\frac{t}{g_s}\,\mathrm{Tr}\,\ln H}
\end{aligned}
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where $\Gamma_q(x) = 1/\prod_{n=0}^{\infty}(1 - x\, q^n)$

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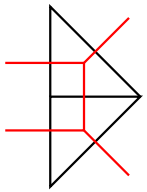

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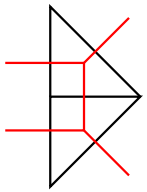

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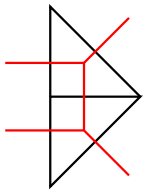

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\end{aligned}
$$

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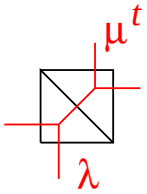
Topological vertex, example 3

Example: Resolved connifold



$$Z_{\lambda,\mu^t} = \frac{\prod_{i<j}[\lambda_i - i - \lambda_j + j]_q \ \prod_{i<j}[\mu_i - i - \mu_j + j]_q}{\prod_{i,j}[\lambda_i - i - \mu_j + j + Q]_q}$$

We write $Q = a - a'$, and $h_i = q^{a+\lambda_i - i + N}$, and $h'_i = q^{a'+\mu_i - i + N}$, $\Delta(h) = \prod_{i<j}(h_i - h_j)$, $\Delta(h,h') = \prod_{i,j}(h_i - h'_j)$:

$$Z_{\lambda,\mu^t} \propto \frac{\Delta(h)\,\Delta(h')}{\Delta(h,h')} = \frac{\prod_{i<j}(h_i - h_j) \ \prod_{i<j}(h'_i - h'_j)}{\prod_{i,j}(h_i - h'_j)}$$

$$= \det\left(\frac{1}{h_i - h'_j}\right)$$

Cauchy determinant formula

Rewrite:

$$\det\left(\frac{1}{h_i - h'_j}\right) = \frac{1}{N!}\sum_{\sigma,\rho}(-1)^{\sigma\rho}\prod_i \frac{1}{h_{\rho(i)} - h'_{\sigma(i)}}$$

Transformation into a matrix integral

Laplace transform: $\frac{1}{h-h'} = \int_0^\infty dr\, \mathrm{e}^{-r(h-h')}$

$$\det\left(\frac{1}{h_i - h'_j}\right)$$

$$= \frac{1}{N!}\sum_{\sigma,\rho}(-1)^{\sigma\rho}\prod_i \int_0^\infty dr_i \mathrm{e}^{-r_i(h_{\rho(i)}-h'_{\sigma(i)})}$$

$$= \frac{1}{N!}\int_0^\infty dr_1\dots dr_N \underbrace{\det\left(\mathrm{e}^{-r_i h_j}\right)}\underbrace{\det\left(\mathrm{e}^{r_i h'_j}\right)}$$

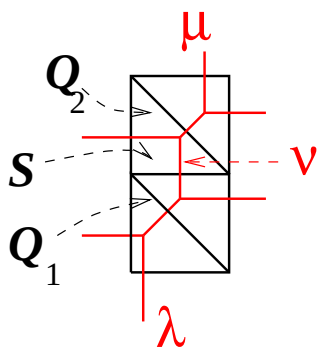
Itzykson – Zuber matrix Integrals

$$= \frac{\Delta(h)\Delta(h')}{N!}\int_0^\infty dr_i \Delta(r)^2 \int_{U(N)^2} dU d\tilde{U}\, \mathrm{e}^{-\mathrm{tr}\, r\,(UhU^\dagger - \tilde{U}h'\tilde{U}^\dagger)}$$

$$= \frac{\Delta(h)\Delta(h')}{N!}\int_{H_N(\mathbb{R}_+)} dR \int_{U(N)^2} dU d\tilde{U}\, \mathrm{e}^{-\mathrm{tr}\, R\,(\overbrace{UhU^\dagger}^{H} - \overbrace{\tilde{U}h'\tilde{U}^\dagger}^{H'})}$$

2 boxes

Example: 2 boxes.
$Q_1 = a - a', Q_2 = a' - a''$.



$$Z = \sum_\nu Z_{\lambda,a;\nu^t,a'}\, \mathrm{e}^{-S\,|\nu|}\, Z_{\nu,a';\mu^t,a''}$$

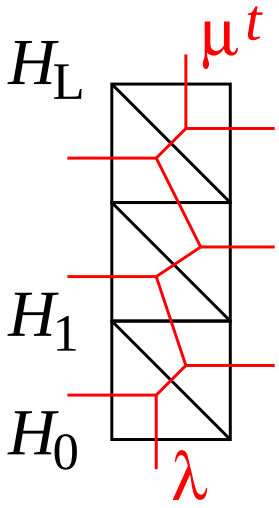
We write $q = \mathrm{e}^{-g_s}$, $h_i(\nu) = q^{\nu_i - i + N + a'}$,
$h = \mathrm{diag}(h_i)$, $\Delta(h) = \prod_{i<j}(h_i - h_j)$, $H = UhU^\dagger$.

$$\begin{aligned} Z &= \sum_\nu \Delta(h(\nu))^2\, \mathrm{e}^{\frac{S}{g_s}\,\mathrm{tr}\,\ln h(\nu)} \int_{H_N(\mathbb{R}_+)} dR\, dU\, \mathrm{e}^{-\mathrm{Tr}\, R(h(\lambda) - Uh(\nu)U^\dagger)} \\ &\qquad \int_{H_N(\mathbb{R}_+)} d\tilde{R}\, d\tilde{U}\ \mathrm{e}^{-\mathrm{Tr}\,\tilde{R}(\tilde{U}h(\nu)\tilde{U}^\dagger - h(\mu))} \\ &= \int_{H_N(\mathcal{C})} \frac{dH\, \mathrm{e}^{\frac{S}{g_s}\,\mathrm{tr}\,\ln H}}{\det f(q^{-a'} H)} \int dR\, d\tilde{R}\, \mathrm{e}^{-\mathrm{Tr}\, R(h(\lambda) - H)}\ \mathrm{e}^{-\mathrm{Tr}\,\tilde{R}(H - h(\mu))} \end{aligned}$$

where $f(x) = 1/\Gamma_q(qx)\,\Gamma_q(1/x)$.

Matrix models and topological strings

Matrix model for the fiducial geometry,
with one column of boxes:



$$
\begin{aligned}
Z \quad = \quad & \int dH_1 \dots dH_{L-1}\, dR_1 \dots dR_L\, \mathrm{e}^{-\sum_i \mathrm{Tr}\, R_i (H_{i-1} - H_i)} \\
& \prod_i \frac{\det\left(\Gamma_q(H_i/q^{a_{i-1}})\right)\ \det\left(\Gamma_q(H_i/q^{a_{i+1}})\right)}{\det\left(\Gamma_q(H_i/q^{a_i})\right)^2} \\
& \prod_i \mathrm{e}^{\frac{S_i+1}{g_s}\, \mathrm{Tr}\, \ln H_i}\ \prod_i \frac{1}{\det\left(f(q^{-a_i} H_i)\right)} \\
& \int dH_0\, \frac{\det\left(\Gamma_q(H_0/q^{a_1})\right)}{\det\left(\Gamma_q(H_0/q^{a_0})\right) \prod_{i=1}^N \det\left(H_0 - q^{\lambda_i - i + N + a_0}\right)} \\
& \int dH_L\, \frac{\det\left(\Gamma_q(H_L/q^{a_{L-1}})\right)}{\det\left(\Gamma_q(H_L/q^{a_L})\right) \prod_{i=1}^N \det\left(H_L - q^{\mu_i - i + N + a_L}\right)}
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Matrix models and topological strings

Matrix model for the fiducial geometry,
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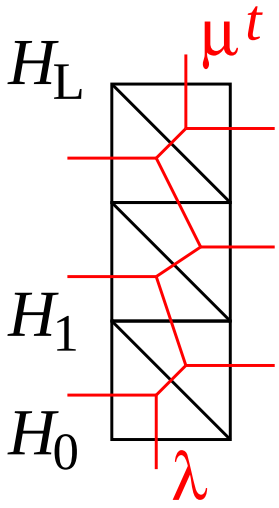

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$$

Matrix models and topological strings

Matrix model for the fiducial geometry,
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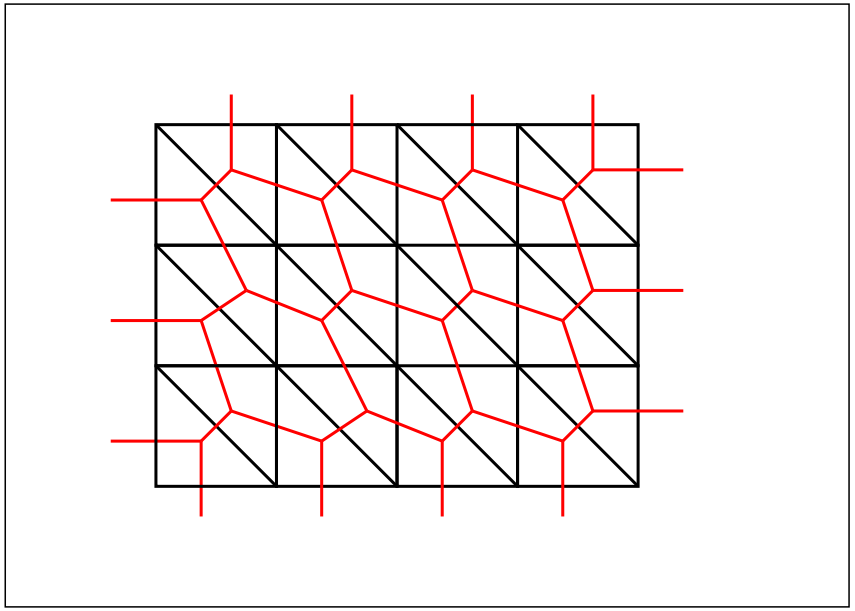

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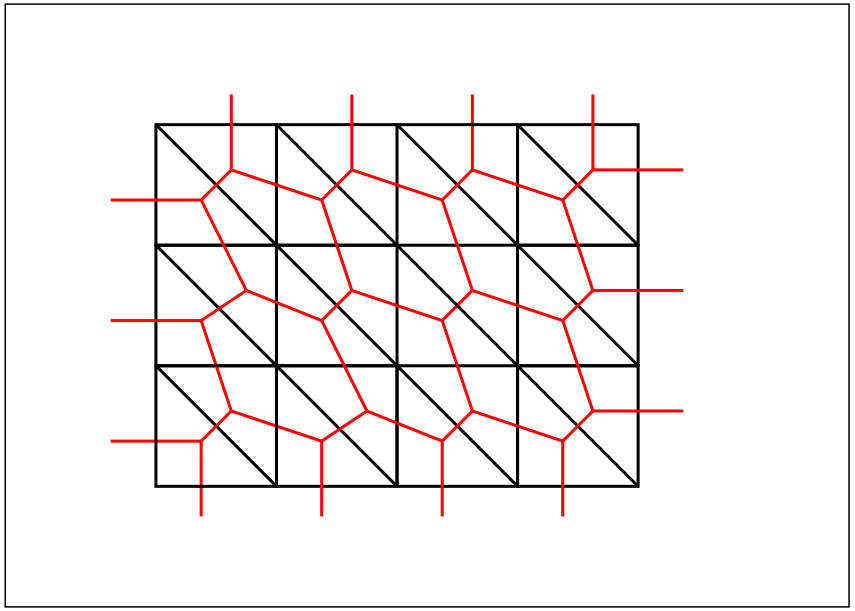
Matrix models and topological strings

Matrix model for the fiducial geometry,
with several columns of boxes:



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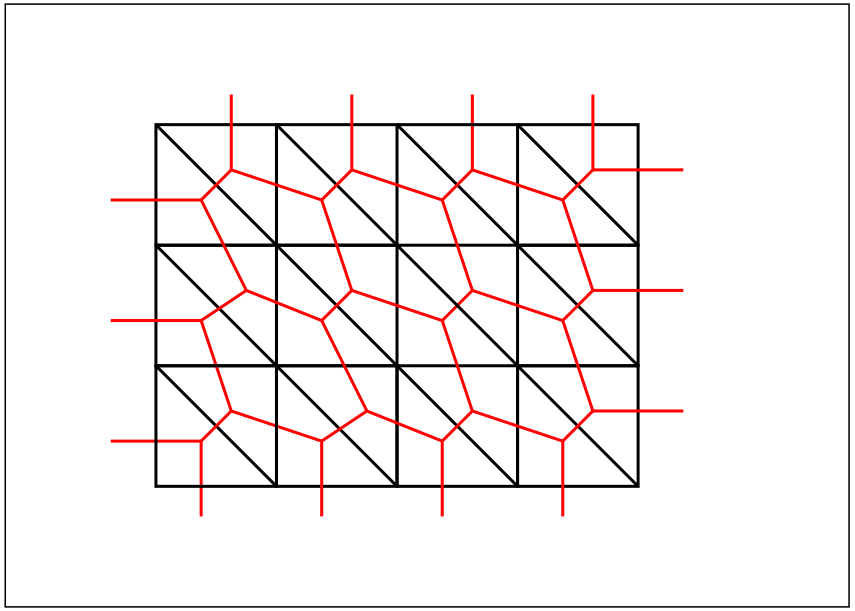

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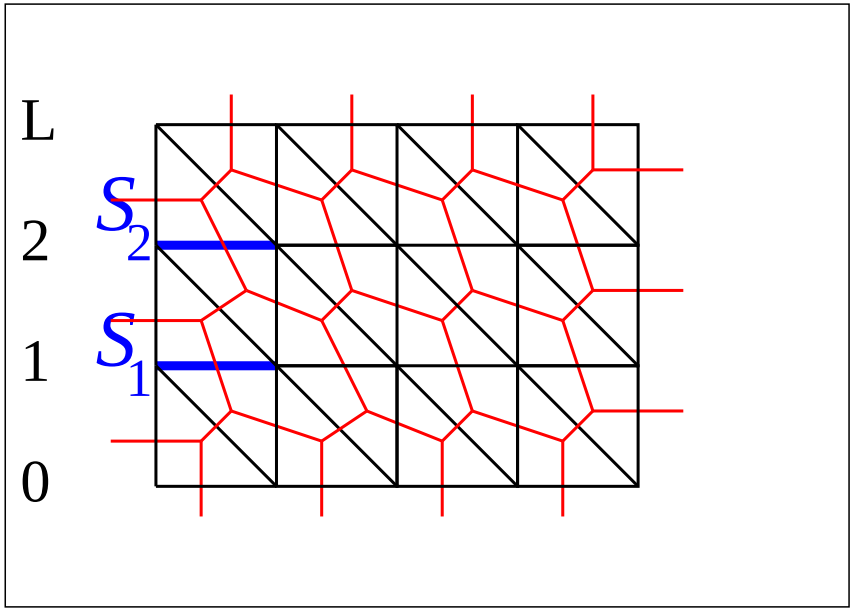

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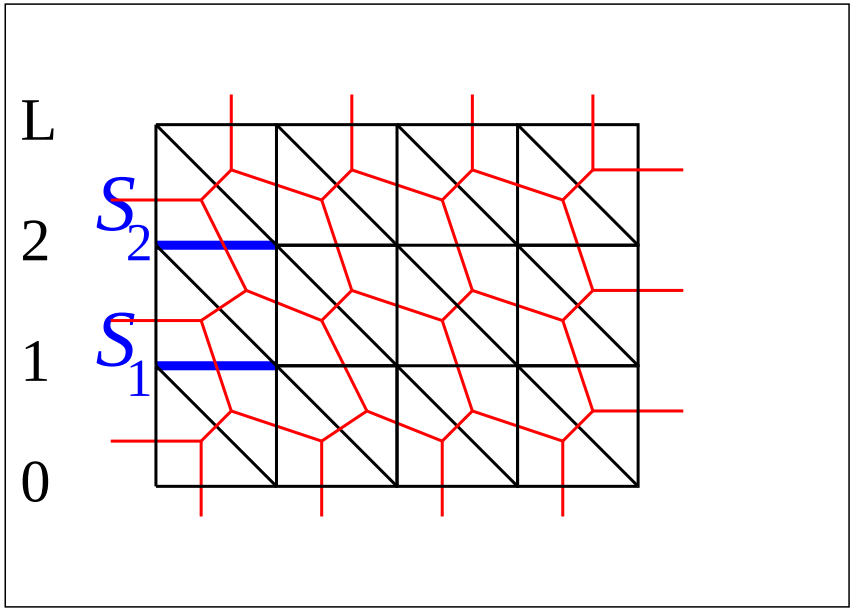

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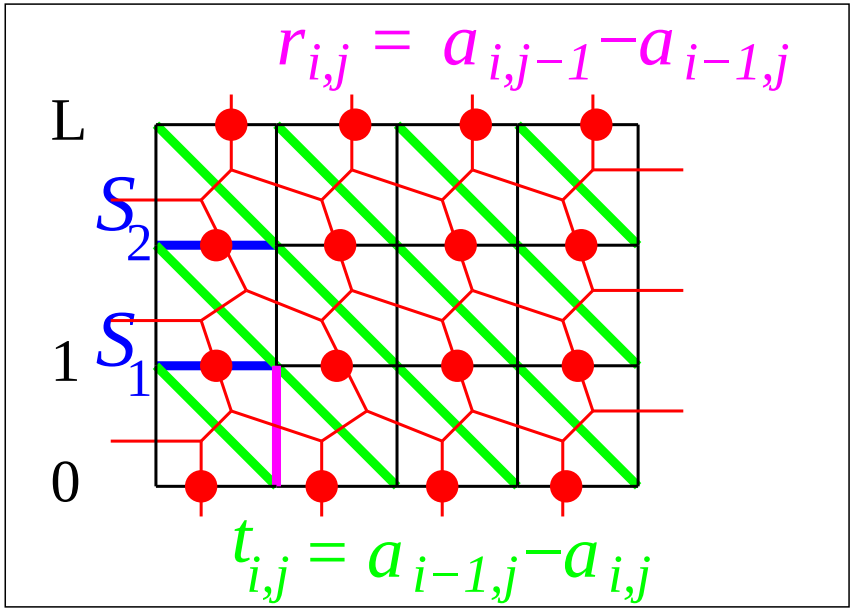

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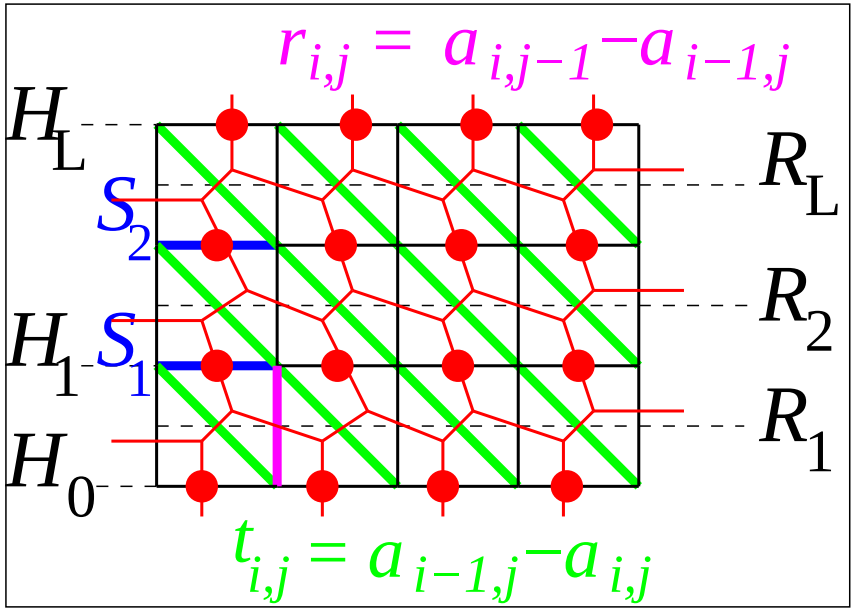

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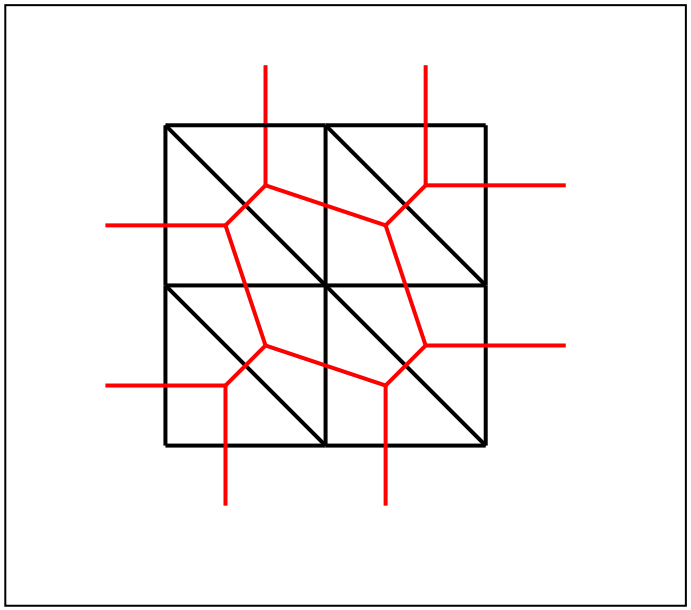
Any CY3 toric geometry can be obtained from the box-fiducial geometry, by flop transitions.

Example: 4 boxes $\longrightarrow$ local $\mathbb{P}^2$



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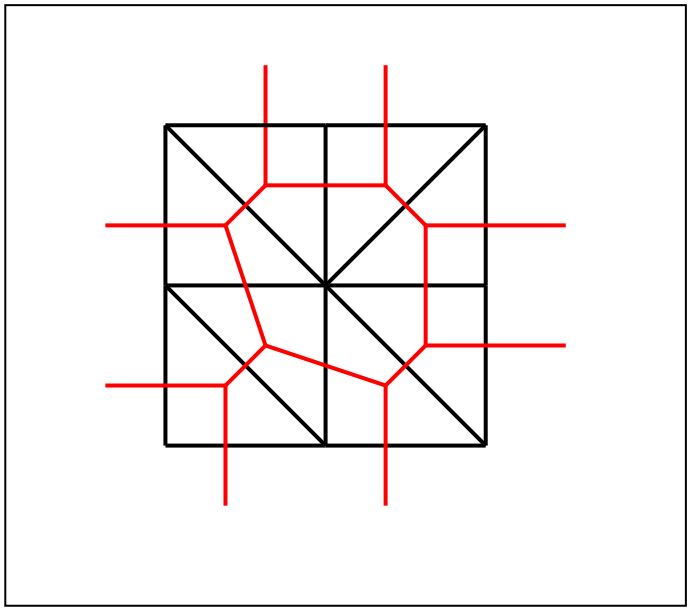

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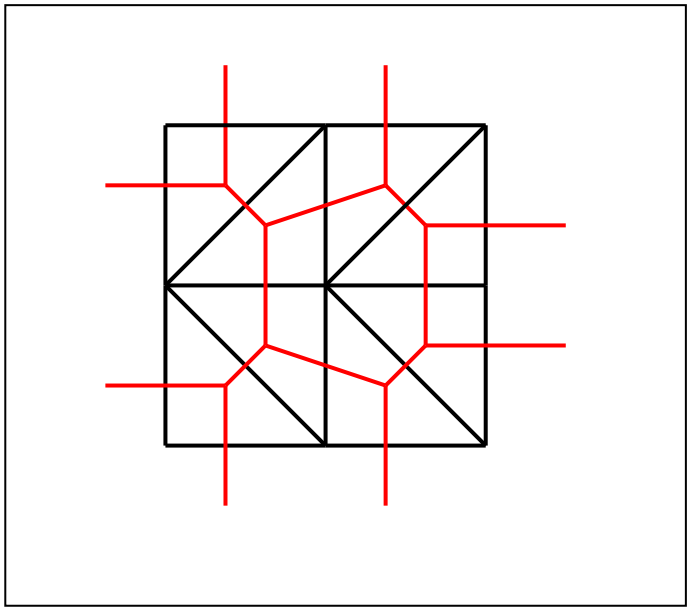

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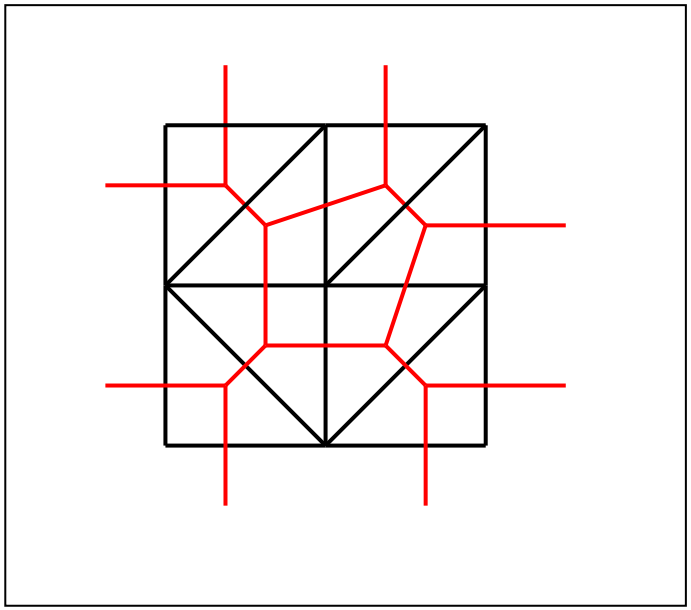

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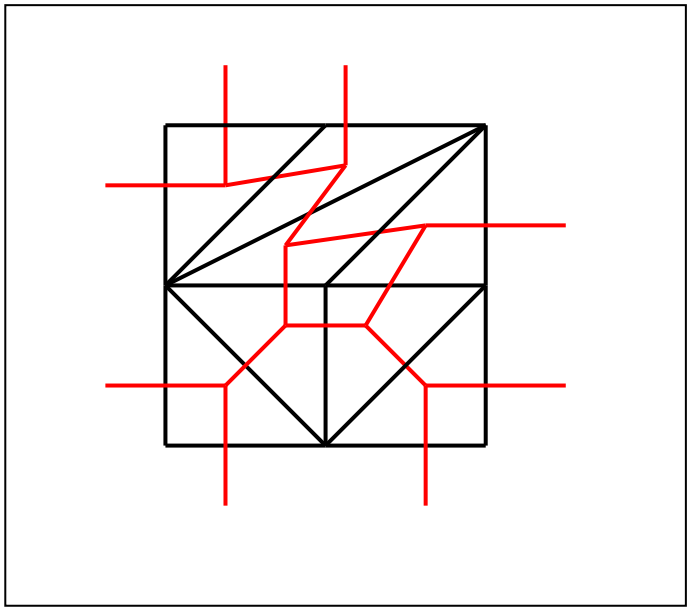

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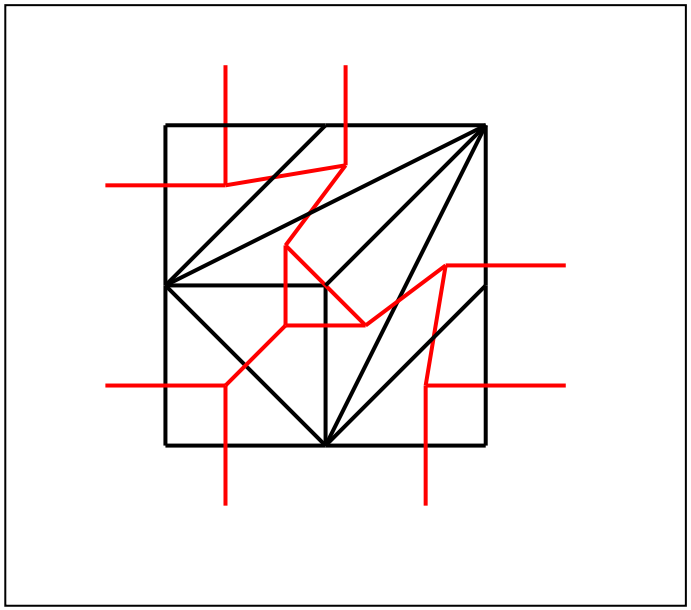

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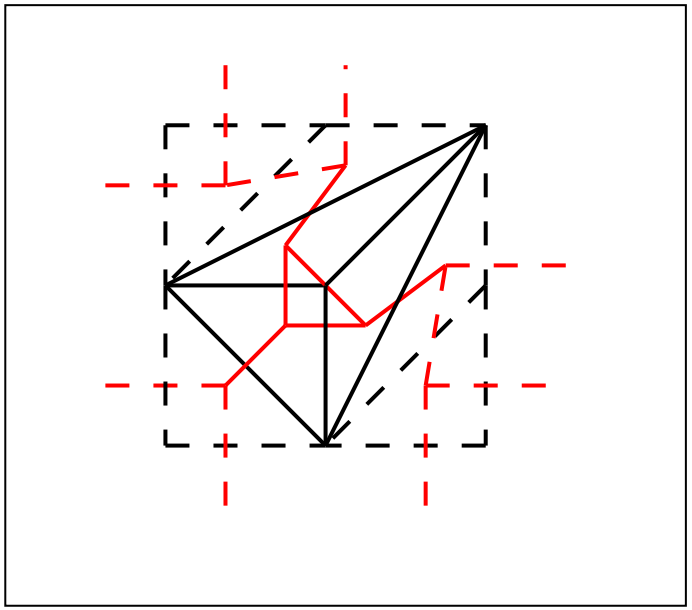

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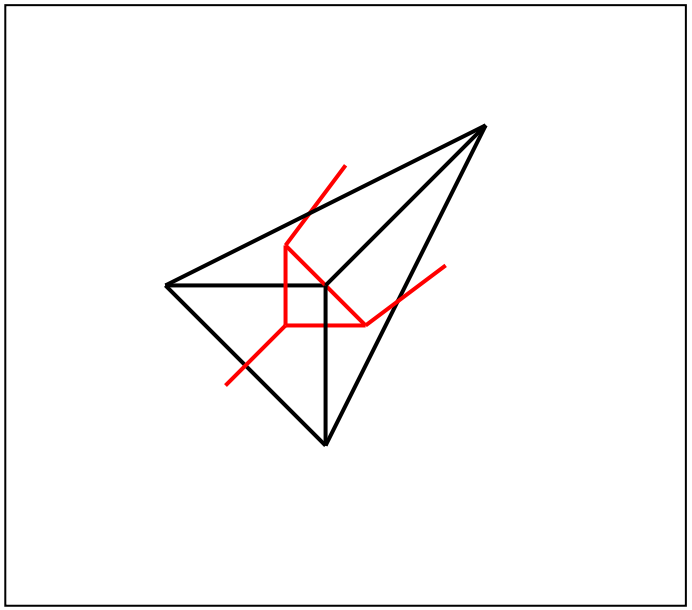

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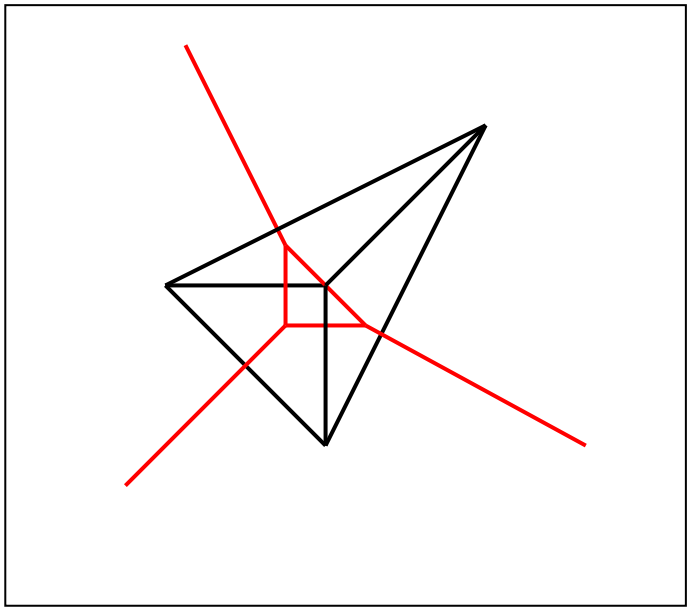

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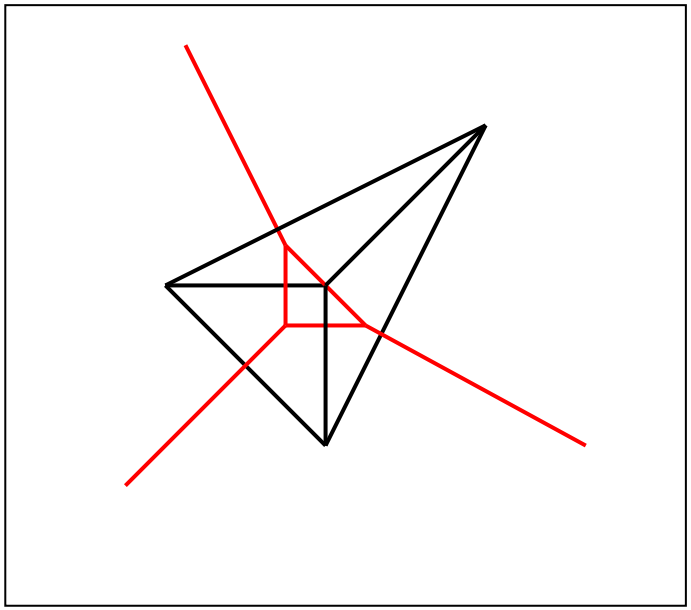
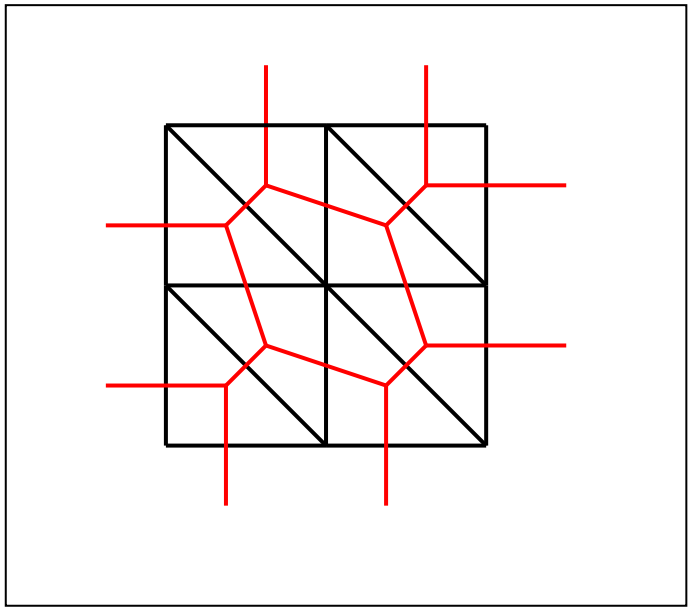


$\longleftrightarrow$



2. Matrix models

Generalities about matrix models

Consider the general chain of matrices:

$$Z = \int dM_1 \dots dM_L \, \mathrm{e}^{-\frac{1}{g_s}\sum_i \mathrm{Tr}\, V_i(M_i)} \, \mathrm{e}^{\frac{1}{g_s}\sum_i c_i \,\mathrm{Tr}\, M_i M_{i+1}}$$

Topological expansion:

$$\ln Z = \sum_g g_s^{2g-2} F_g$$

Question: how to compute F_g's ?

Method: write and solve loop equations

Generalities about loop equations

General chain of matrices:

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Loop equations = integration by parts = Schwinger-Dyson eqs.
$\rightarrow$ give relationships among expectation values.

Example: $\forall i$, $\quad \langle \mathrm{Tr}\, V_i'(M_i) \rangle = c_i \langle \mathrm{Tr}\, M_{i+1} \rangle + c_{i-1} \langle \mathrm{Tr}\, M_{i-1} \rangle$

Resolvents: $W_i(x) = \langle \mathrm{Tr}\, 1/(x - M_i) \rangle = \sum_g g_s^{2g-1} W_{1,i}^{(g)}(x)$ is such that:
$W_{1,i}^{(0)}(x)$ satisfies a non-linear equation,
$W_{1,i}^{(g)}(x)$ with $g \geq 1$ satisfy linear equations.

Generalities about loop equations

More generally: n-point correaltion functions for M_i:

$$\begin{aligned} W_{n,i}(x_1,\dots,x_n) &= \left\langle \mathrm{Tr}\,\frac{1}{x_1-M_i}\,\mathrm{Tr}\,\frac{1}{x_2-M_i}\dots\mathrm{Tr}\,\frac{1}{x_n-M_i}\right\rangle_c \\ &= \sum_g g_s^{2g-2+n} W_{n,i}^{(g)}(x_1,\dots,x_n) \end{aligned}$$

Theorem (Topological recursion E. 2004, Chekhov + E. + Orantin + Prats-Ferrer 2004 - 2009)

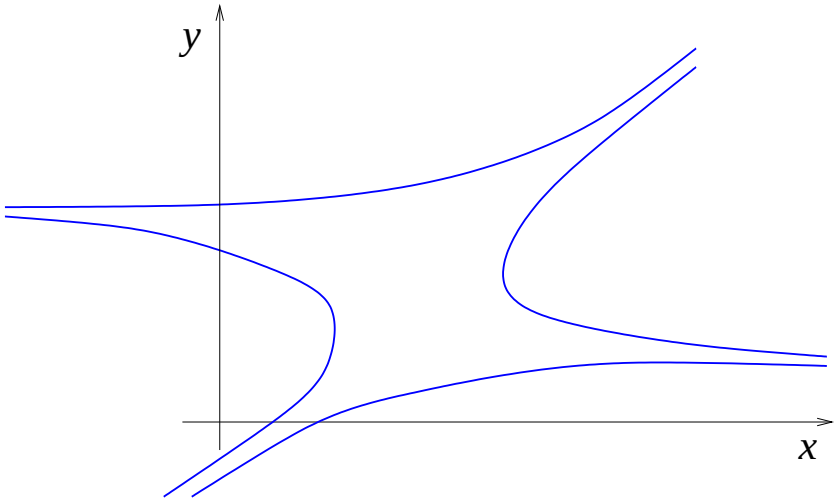
For all chain of matrices: Loop equations imply that:

- $W_{1,i}^{(0)}(x)$ *satisfies a non-linear equation = spectral curve of* M_i,
- $W_{n,i}^{(g)}(x_1,\dots,x_n)$ *with* $2g-2+n\geq 0$ *satisfy a recursive set of linear equations. The solution of this recursion is the topological recursion.*
- *Then, knowing* $W_{n,i}^{(g)}$ *determines* F_g: $\ln Z = \sum_g g_s^{2g-2}F_g$.

Spectral curve and topological recursion

The spectral curve $y_i = W_{1,i}^{(0)}(x)$ can be found (for instance) from saddle point approximation.

The function $x \mapsto y = W_{1,i}^{(0)}(x)$ defines a Riemann surface embedded into $\mathbb{C} \times \mathbb{C}$ =plane curve.



- Bergman kernel $B(z_1, z_2)$ = fundamental 2-form of the second kind on the spectral curve (derivative of the Green function) $W_{2,i}^{(0)}(x_1, x_2)dx_1 dx_2 = B(x_1, x_2) - \frac{dx_1\, dx_2}{(x_1 - x_2)^2}$
- Recursion kernel $K(z_0, z) = \frac{\int_{z'=\bar{z}}^{z} B(z_0, z')}{2(W_1^{(0)}(z) - W_1^{(0)}(\bar{z}))\, dx(z)}$

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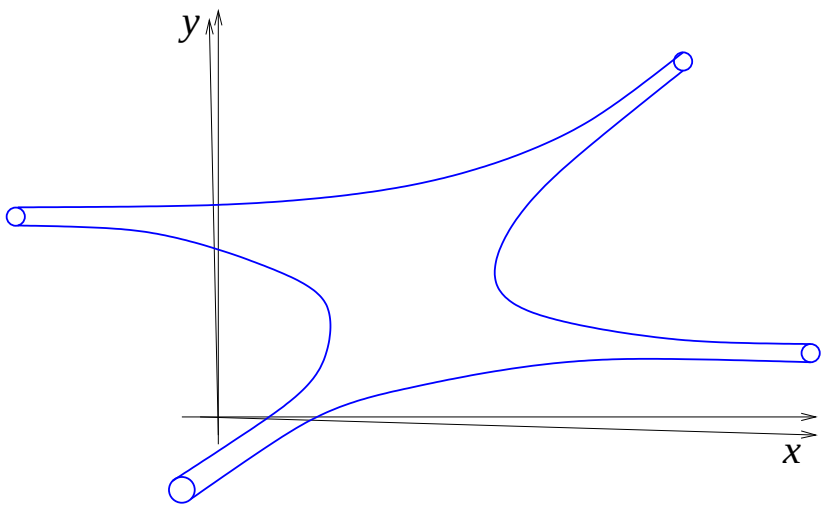

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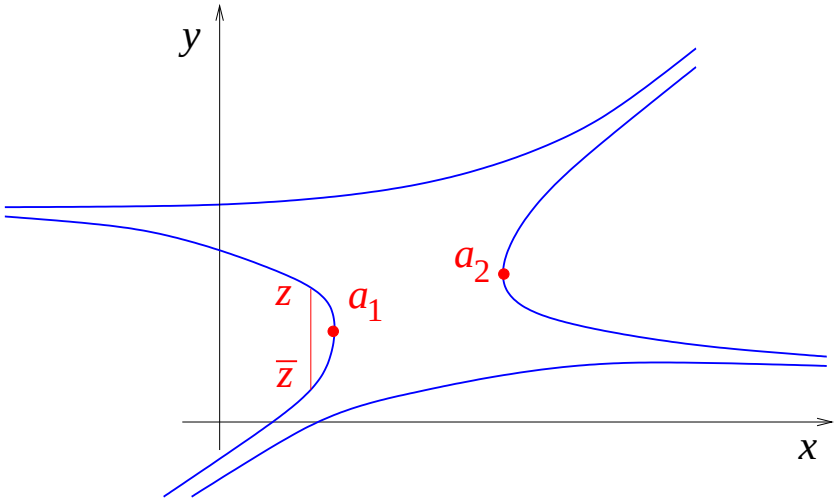

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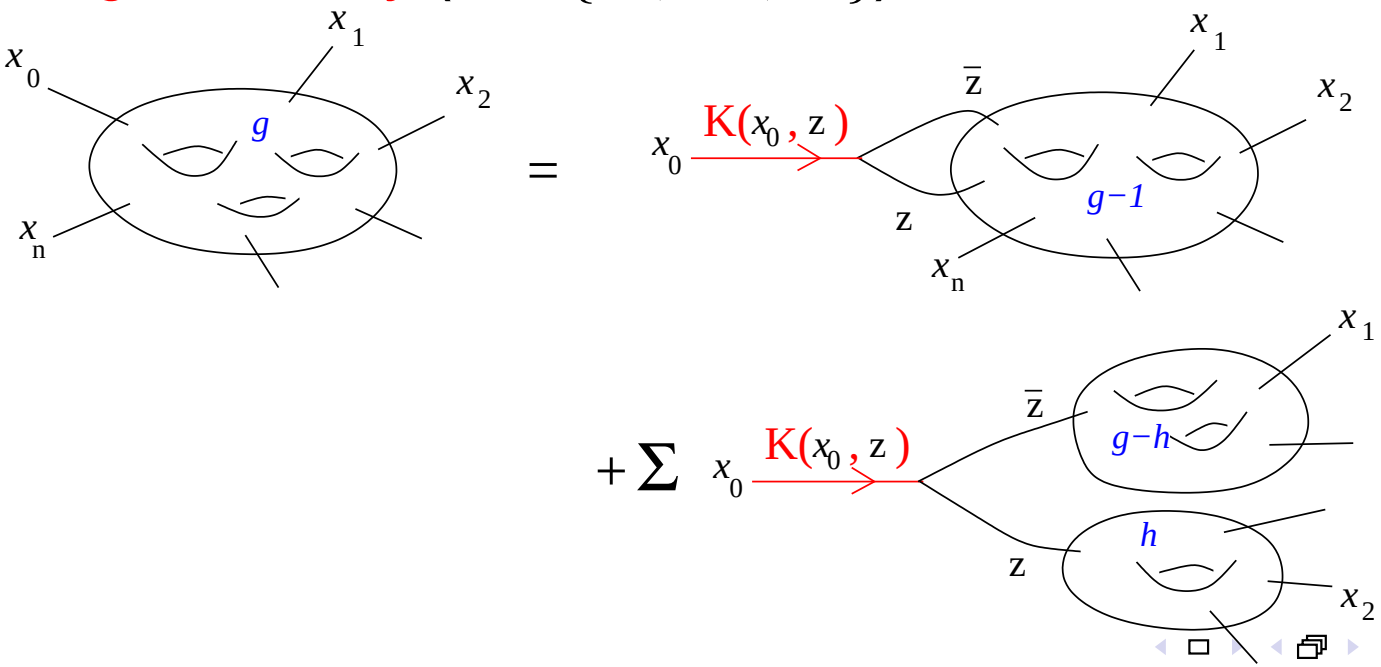
Spectral curve and topological recursion

When the spectral curve is known, then, all higher order correlation functions are found by the "topological recursion"

$$W^{(g)}_{n+1}(x_0,J) = \sum_i \operatorname*{Res}_{z\to a_i} K(x_0,z)\Big[W^{(g-1)}_{n+2}(z,\bar z,J) + \sum_h \sum_{I\subset J}{}' W^{(h)}_{1+\#I}(z,I)\, W^{(g-h)}_{1+n-\#I}(\bar z, J\setminus I)\Big]$$

Diagramatically ($J=\{x_1,\dots,x_n\}$):

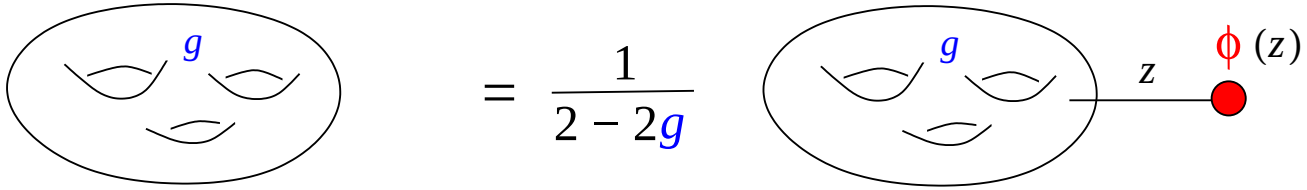


Spectral curve and topological recursion

When the the $W_n^{(g)}$'s are known, then, the F_g's are given by:

$$\forall g \geq 2 \qquad F_g = \frac{1}{2-2g} \sum_i \operatorname*{Res}_{x \to a_i} \Phi(x)\, W_1^{(g)}(x)$$

where $d\Phi/dx = y = W_1^{(0)}(x)$.
Diagramatically:



Property: $F_g =$ symplectic invariant.

Remark: all spectral curves $y_i(x) = W_{1,i}^{(0)}(x)$ are symplectically equivalent. $\Rightarrow F_g(\{y_{i+1}(x)\}) = F_g(\{y_i(x)\})$.
F_g is independent of which matrix M_i we choose.

3. Remodeling the B-model

Remodeling the B-model

Partition function of topological strings on a toric CY3 $\mathfrak{X}$:

$$\ln Z = \sum_g g_s^{2g-2} F_g$$

F_g = topological string amplitude of closed worldsheets of genus g.
$W_{n,i}^{(g)}$ = topological string amplitude of "open" worldsheets of genus g with "n boundaries on a brane of type i".

Conjecture (Remodeling the B-model, Mariño 2006, BKMP 2008)

- *The spectral curve $y = W_{1,i}^{(0)}(x)$ is (modulo symplectic transformation) the mirror curve of the target space $\mathfrak{X}$.*
- *The higher $W_{n,i}^{(g)}$ and F_g are obtained by the topological recursion.*

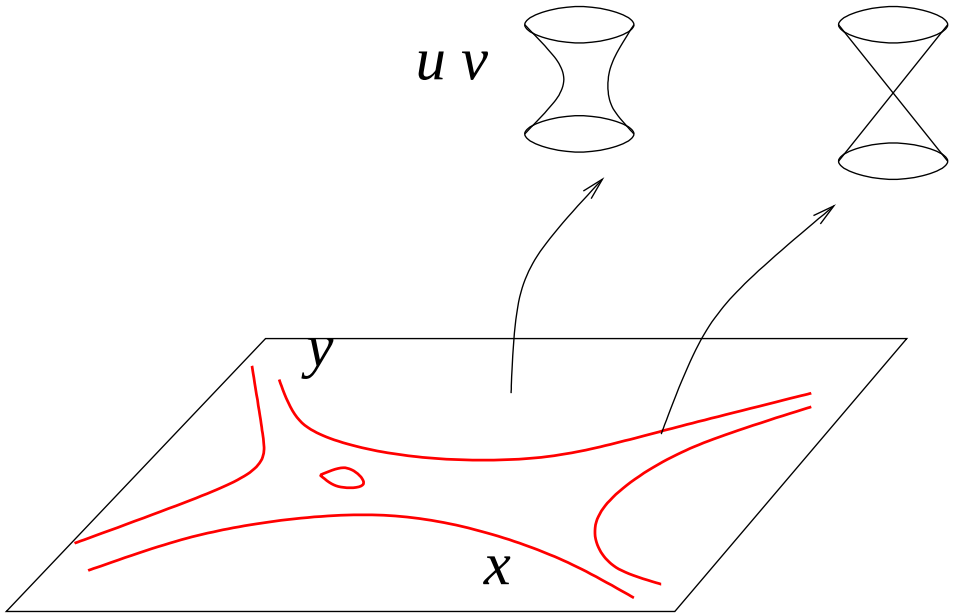
Mirror curve

Mirror symmetry: CY $\mathfrak{X} \longrightarrow$ CY $\hat{\mathfrak{X}}$

If $\mathfrak{X}$ is toric, then $\hat{\mathfrak{X}}$ is an hyperbolic bundle over a complex plane curve:

$$\hat{\mathfrak{X}} : \qquad H(\mathrm{e}^x, \mathrm{e}^y) = uv$$

where $H =$ polynomial.

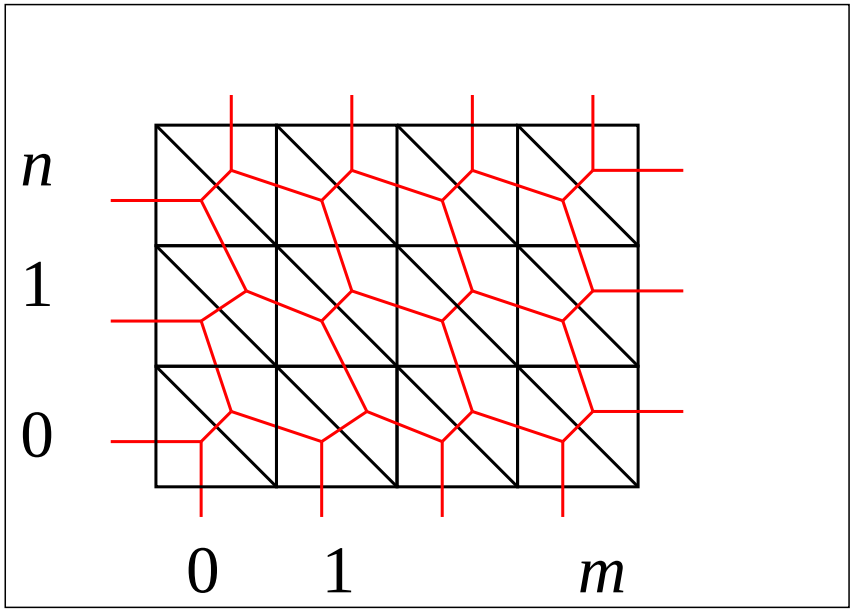


Fibers degenerate over the mirror curve: $\qquad H(\mathrm{e}^x, \mathrm{e}^y) = 0$

Mirror curve

For the box geometry of size $n \times m$, mirror curve: $H(e^x, e^y) = 0$.
H is a polynomial of degree n, m:

$$H(X,Y) = \sum_{i=0}^{n} \sum_{j=0}^{m} H_{i,j} X^j Y^i$$



Coefficients $H_{i,j}$ determined by periods

$$\oint_{\mathcal{A}_{i,j}} y\, dx = \oint_{\mathcal{A}_{i,j}} \ln Y \frac{dX}{X} = t_{i,j} = \text{Kaehler moduli}$$

Remark: # branchpoints = # vertices.

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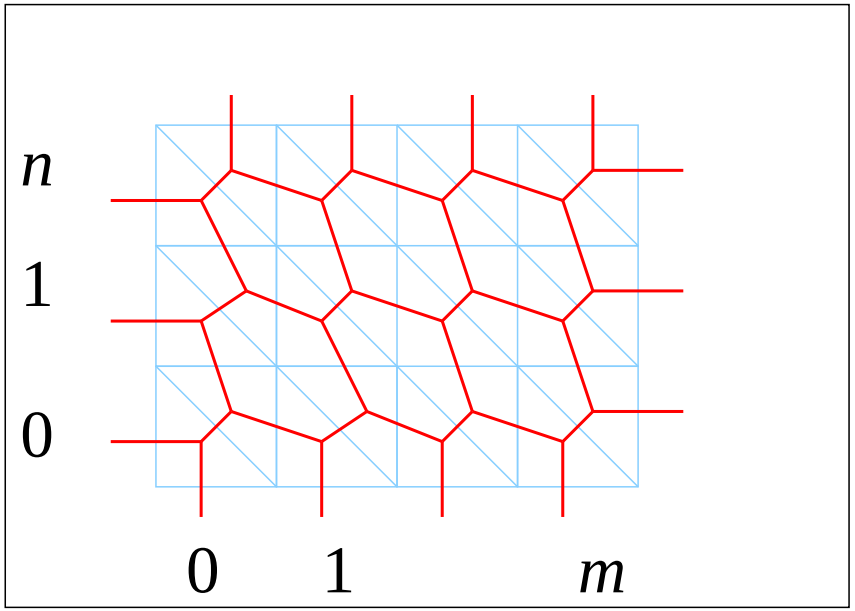

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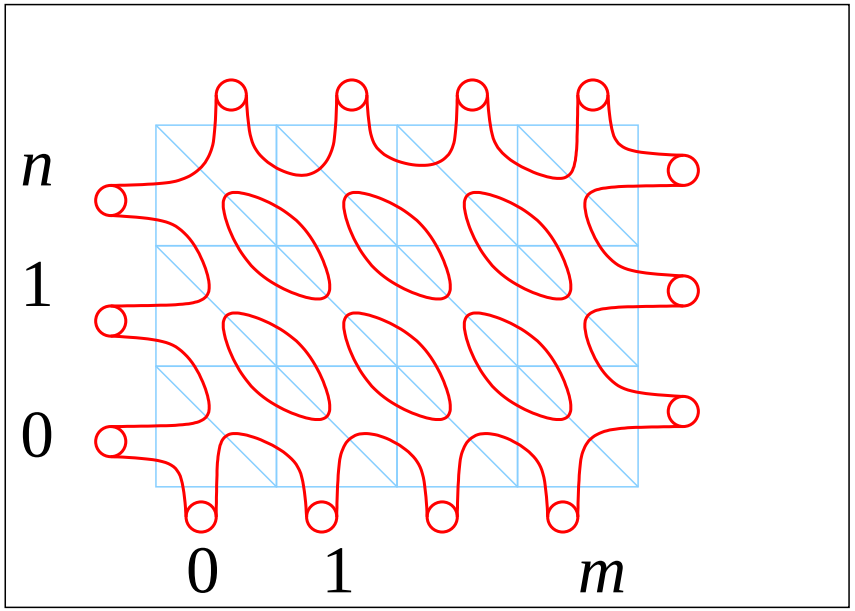

Coefficients $H_{i,j}$ determined by periods

$$\oint_{\mathcal{A}_{i,j}} y\, dx = \oint_{\mathcal{A}_{i,j}} \ln Y \, \frac{dX}{X} = t_{i,j} = \text{Kaehler moduli}$$

Remark: # branchpoints = # vertices.

Mirror curve

For the box geometry of size $n \times m$, mirror curve: $H(e^x, e^y) = 0$.
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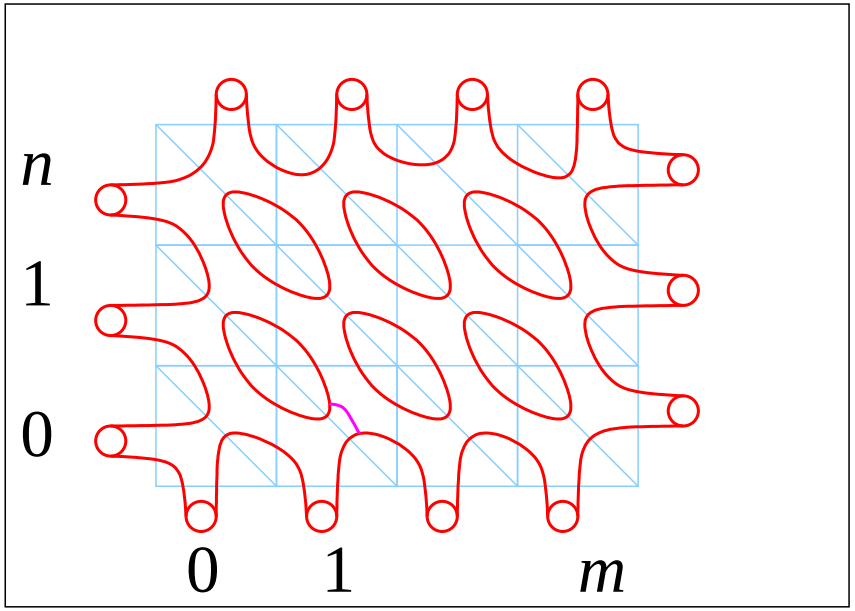

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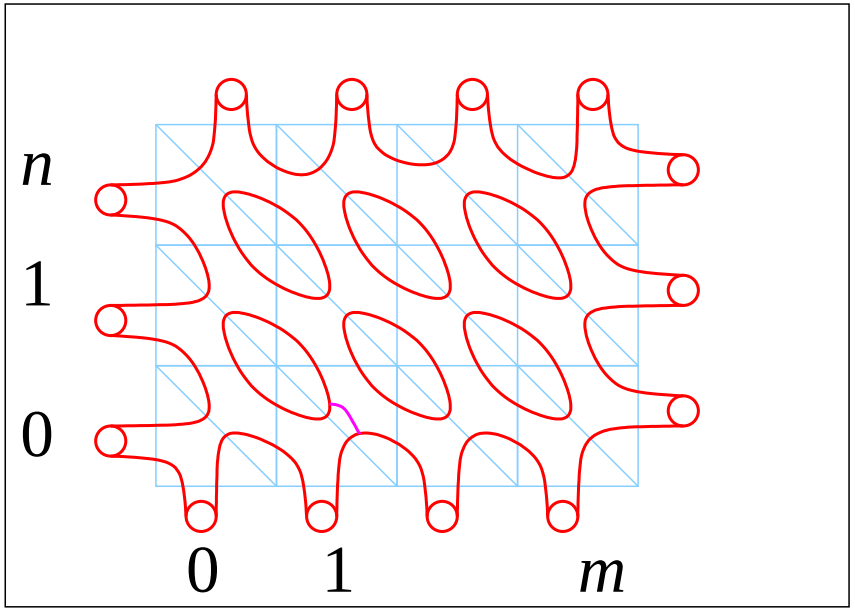

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4. Matrix model's spectral curve

Matrix model's spectral curve

Towards a proof: The mirror curve satisfies 1), 2), 3) !

$$Z = \int dM_1 \dots dM_L \, \mathrm{e}^{-\frac{1}{g_s}\sum_i \mathrm{Tr}\, V_i(M_i)} \, \mathrm{e}^{\frac{1}{g_s}\sum_i c_i \,\mathrm{Tr}\, M_i M_{i+1}}$$

$$M_i \in H_N(\prod_j \gamma_{i,j}^{n_{i,j}})$$

Recipe: find a complex curve $\mathcal{C}$, and functions $x_i(z)$: $\mathcal{C} \to \mathbb{C}$, $i = 0, \dots, L+1$, satisfying:

- 1) equations of motion : $V_i'(x_i(z)) = c_{i-1}x_{i-1}(z) + c_i x_{i+1}(z)$.
- 2) filling fraction condition : $\frac{c_i}{2\pi \mathrm{i}} \oint_{\mathcal{A}_{i,j}} x_{i+1}\, dx_i = n_{i,j}/N g_s$.
- 3) endpoints : x_{i+1} has simple pole at x_i =endpoint for M_i.
- 4) Find the choice which minimizes a certain action $\mathcal{S}_0$.

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5. Conclusion

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- We can construct an explicit matrix model for topological strings in any toric geometry.
- This gives the possibility to use all the toolbox of matrix models: integrability, orthogonal polynomials, loop equations, topological recursion, wall crossing (Riemann-Hilbert), etc...
- New light on mirror symmetry. The mirror curve and mirror map emmerge from the matrix model.
- Non-perturbative definition of topological string. The matrix model is non-perturbative. All instantons resummations are known $\rightarrow$ background independence. Also $\rightarrow$ holomorphic anomaly equations [BCOV], integrability, quantum Riemann surfaces. Maybe link with AGT. Can we see other features of string theory in the matrix model ? Dualities ?