Random Matrices and topological strings

Bertrand Eynard, IPHT CEA Saclay, CERN
GENEZISS String Theory Meeting, EPFL, Lausane
based on collaborations with A. Kashani-Poor, O. Marchal,

november 26th 2010
Outline

1. Introduction Topological strings
   - the topological vertex method
   - rewriting as a matrix integral

2. Generalities about matrix integrals
   - loop equations, topological recursion, spectral curve

3. remodeling the B-model, mirror symmetry
   - the Mariño-BKMP conjecture - the mirror curve

4. Matrix model’s spectral curve
   - finding the mirror curve from the matrix model

5. Conclusion
1. Topological strings
Topological strings

- Type IIA, topological string theory in a target space $\mathcal{X}$, which we choose to be a CY 3-fold, with Toric symmetry.

- Useful for: some low energy limit of string theory, captures some exact features of string theory. Applications to mathematics: algebraic geometry, Gromov-Witten invariants. Application to knot invariants, equivalence with Chern-Simons theory.
Toric symmetry and topological vertex

CY3-fold can be obtained by gluing $\mathbb{C}^3$ patches.

$\mathbb{C}^3$ has a canonical $T^3$ torus action $z_i \rightarrow z_i e^{i\theta_i}$, which degenerates when $|z_i|^2 = 0$. The lines where two toric symmetries degenerate draw a diagram in $\mathbb{R}^3$, project it to $\mathbb{R}^2 = \text{Toric diagram}$.

String amplitudes are computed by "gluing" $\mathbb{C}^3$ open string amplitudes, and summing over gluing moduli: $\rightarrow$ topological vertex formula [AKMV 2003].
All A-model topological string theory partition functions in toric CY3 target spaces, can be written as sums over partitions, in a way encoded by the toric diagram.

Goal: • rewrite it as a matrix integral.
  • see how mirror symmetry emerges from it, remodelling the B-model, topological recursion, integrability, holomorphic anomaly, non-perturbative part, ...

Bertrand Eynard, IPHT CEA Saclay, CERN GENEZISS String The Random Matrices and topological strings
Example 1: Nekrasov’s SW $SU(1)$ partition function

\[ Z_{4d} = \sum_\lambda \left( \frac{\dim \lambda}{|\lambda|!} \right)^2 e^{-t|\lambda|} \]

\[ = \sum_{h_1 \geq \cdots \geq h_N \geq 0} \frac{\Delta(h)^2}{\prod_i h_i!^2} \prod_i e^{-t \sum h_i} \quad \Delta(h) = \prod_{i > j} (h_i - h_j) \]

\[ = \frac{1}{N!} \sum_{h_1, \ldots, h_N \geq 0} \frac{\Delta(h)^2}{\prod_i h_i!^2} \prod_i e^{-t \sum h_i} \]

\[ = \frac{1}{N!} \oint_{C_N} dh_1 \ldots dh_N \Delta(h)^2 \prod_i \frac{\Gamma(-h_i) e^{i\pi h_i}}{\Gamma(h_i + 1)} e^{-t \sum h_i} \]

\[ = \frac{1}{N!} \int_{H_N(C)} dH \frac{\det(\Gamma(-H)) e^{i\pi \text{Tr} H}}{\det(\Gamma(1 + H))} e^{-t \text{Tr} H} \quad H = U h U^\dagger \]
Example 1: Nekrasov’s SW $SU(1)$ partition function

\[ Z_{4d} = \sum_{\lambda} \left( \frac{\text{dim} \lambda}{|\lambda|!} \right)^2 e^{-t|\lambda|} \]

\[ = \sum_{h_1,\ldots,h_N \geq 0} \frac{\Delta(h)^2}{\prod_i h_i!^2} \prod_i e^{-t \sum_i h_i} \]

\[ = \frac{1}{N!} \sum_{h_1,\ldots,h_N \geq 0} \frac{\Delta(h)^2}{\prod_i h_i!^2} \prod_i e^{-t \sum_i h_i} \]

\[ = \frac{1}{N!} \int_{C_N} dh_1 \ldots dh_N \Delta(h)^2 \prod_i \frac{\Gamma(-h_i) e^{i\pi h_i}}{\Gamma(h_i + 1)} e^{-t \sum_i h_i} \]

\[ = \frac{1}{N!} \int_{H_N(C)} dH \frac{\det(\Gamma(-H)) e^{i\pi \text{Tr} H}}{\det(\Gamma(1 + H))} e^{-t \text{Tr} H} \]

\[ \Delta(h) = \prod_{i > j} (h_i - h_j) \]

\[ h_i = \lambda_i - i + N \]
Example 1: Nekrasov's SW $SU(1)$ partition function

$$Z_{4d} = \sum_{\lambda} \left( \frac{\text{dim} \lambda}{|\lambda|!} \right)^2 e^{-t|\lambda|}$$

**note** $h_i = \lambda_i - i + N$

$$= \sum_{h_1 > \ldots > h_N \geq 0} \frac{\Delta(h)^2}{\prod_i h_i!^2} \prod_i e^{-t \sum_i h_i}$$

$$\Delta(h) = \prod_{i > j} (h_i - h_j)$$

$$= \frac{1}{N!} \sum_{h_1, \ldots, h_N \geq 0} \frac{\Delta(h)^2}{\prod_i h_i!^2} \prod_i e^{-t \sum_i h_i}$$

$$= \frac{1}{N!} \int_{\mathbb{C}^N} dh_1 \ldots dh_N \Delta(h)^2 \prod_i \frac{\Gamma(-h_i) e^{i\pi h_i}}{\Gamma(h_i + 1)} e^{-t \sum_i h_i}$$

$$= \frac{1}{N!} \int_{H_N(\mathbb{C})} dH \frac{\det(\Gamma(-H))}{\det(\Gamma(1 + H))} e^{i\pi \text{Tr} H} e^{-t \text{Tr} H}$$

$H = UhU^\dagger$
Example 1: Nekrasov’s SW $SU(1)$ partition function

\[ Z_{4d} = \sum_{\lambda} \left( \frac{\dim \lambda}{|\lambda|!} \right)^2 e^{-t|\lambda|} \]

\[ = \sum_{h_1 > \ldots > h_N \geq 0} \frac{\Delta(h)^2}{\prod_i h_i!^2} \prod_i e^{-t \sum_i h_i} \Delta(h) = \prod_{i > j} (h_i - h_j) \]

\[ = \frac{1}{N!} \sum_{h_1, \ldots, h_N \geq 0} \frac{\Delta(h)^2}{\prod_i h_i!^2} \prod_i e^{-t \sum_i h_i} \]

\[ = \frac{1}{N!} \oint_{C_N} dh_1 \ldots dh_N \Delta(h)^2 \prod_i \frac{\Gamma(-h_i) e^{i\pi h_i}}{\Gamma(h_i + 1)} e^{-t \sum_i h_i} \]

\[ = \frac{1}{N!} \int_{H_N(C)} dH \frac{\det (\Gamma(-H))}{\det (\Gamma(1 + H))} e^{i\pi \text{Tr} H} e^{-t \text{Tr} H} \quad H = UhU^\dagger \]
Example 1: Nekrasov’s SW $SU(1)$ partition function

\[ Z_{4d} = \sum_{\lambda} \left( \frac{\dim \lambda}{|\lambda|!} \right)^2 e^{-t|\lambda|} \]

\[ = \sum_{h_1 > \ldots > h_N \geq 0} \frac{\Delta(h)^2}{\prod_i h_i!^2} \prod_i e^{-t \sum_i h_i} \]

\[ = \frac{1}{N!} \sum_{h_1, \ldots, h_N \geq 0} \frac{\Delta(h)^2}{\prod_i h_i!^2} \prod_i e^{-t \sum_i h_i} \]

\[ = \frac{1}{N!} \oint_{C^N} dh_1 \ldots dh_N \Delta(h)^2 \prod_i \frac{\Gamma(-h_i) e^{i\pi h_i}}{\Gamma(h_i + 1)} e^{-t \sum_i h_i} \]

\[ = \frac{1}{N!} \int_{H_N(C)} dH \frac{\det(\Gamma(-H)) e^{i\pi \text{Tr} H}}{\det(\Gamma(1 + H))} e^{-t \text{Tr} H} \]

\[ H = U h U^\dagger \]
Topological vertex, example 2

in 5-dimension, with radius \( q = e^{-g_s} \), and \( h_i = q^{\lambda_i - i + N} \):

\[
Z_{5d} = \sum_{\lambda} \left( \frac{\text{dim}_{q, \lambda}}{[|\lambda|] q!} \right)^2 e^{-t |\lambda|} = \\
\frac{1}{N!} \sum_{h_1, \ldots, h_N} \frac{\prod_{i<j} (h_i - h_j)^2}{\prod_i h_i^{N-1} \Gamma_q(q h_i)^2} e^{-\frac{1}{2g_s} \sum_i (\ln h_i)^2} e^{\frac{t}{g_s} \sum_i \ln h_i} = \\
\frac{1}{N!} \int_{H_{N(C)}} dH \frac{\det (\Gamma_q(1/H))}{\det (\Gamma_q(q H))} e^{\frac{i \pi}{g_s} \text{Tr} \ln H} e^{\frac{t}{g_s} \text{Tr} \ln H} \text{ where } \Gamma_q(x) = 1 / \prod_{n=0}^{\infty} (1 - x q^n)
\]
in 5-dimension, with radius $q = e^{-g_s}$, and $h_i = q^{\lambda_i - i + N}$:

$$Z_{5d} = \sum_{\lambda} \left( \frac{\dim_q \lambda}{[|\lambda|]_q!} \right)^2 e^{-t |\lambda|}$$

$$= \frac{1}{N!} \sum_{h_1, \ldots, h_N} \frac{\prod_{i<j} (h_i - h_j)^2}{\prod_i h_i^{N-1} \Gamma_q(q h_i)^2} e^{-\frac{1}{2g_s} \sum_i (\ln h_i)^2} e^{\frac{t}{g_s} \sum_i \ln h_i}$$

$$= \frac{1}{N!} \int_{H_N(C)} dH \frac{\det (\Gamma_q(1/H))}{\det (\Gamma_q(qH))} e^{\frac{i\pi}{g_s} \operatorname{Tr} \ln H} e^{\frac{t}{g_s} \operatorname{Tr} \ln H}$$

where $\Gamma_q(x) = 1 / \prod_{n=0}^{\infty} (1 - x q^n)$
in 5-dimension, with radius $q = e^{-g_s}$, and $h_i = q^{\lambda_i - i + N}$:

$$Z_{5d} = \sum_{\lambda} \left( \frac{\dim_q \lambda}{[\lambda]_q} \right)^2 e^{-t|\lambda|}$$

$$= \frac{1}{N!} \sum_{h_1, \ldots, h_N} \frac{\prod_{i<j} (h_i - h_j)^2}{\prod_i h_i^{N-1/2} \Gamma_q(q h_i)^2} e^{-\frac{1}{2g_s} \sum_i (\ln h_i)^2} e^{\frac{t}{g_s} \sum \ln h_i}$$

$$= \frac{1}{N!} \int_{H_{N(C)}} dH \frac{\det (\Gamma_q(1/H))}{\det (\Gamma_q(qH))} e^{\frac{i\pi}{g_s} \text{Tr} \ln H} e^{\frac{t}{g_s} \text{Tr} \ln H}$$

where $\Gamma_q(x) = 1 / \prod_{n=0}^{\infty} (1 - x q^n)$
Topological vertex, example 2

in 5-dimension, with radius \( q = e^{-g_s} \), and \( h_i = q^{\lambda_i - i + N} \):

\[
Z_{5d} = \sum_{\lambda} \left( \frac{\dim_q \lambda}{[\lambda]_q !} \right)^2 e^{-t |\lambda|} = \frac{1}{N!} \sum_{h_1,\ldots,h_N} \frac{\prod_{i<j} (h_i - h_j)^2}{\prod_i h_i ^{N-1/2} \Gamma_q (q h_i)^2} \frac{1}{\det (\Gamma_q (1/H))} \frac{1}{\det (\Gamma_q (qH))} \frac{e^{-1/g_s} \sum_i (\ln h_i)^2}{e^{t/g_s} \sum_i \ln h_i \Gamma_q (x) = 1 \prod_{n=0}^\infty (1 - x q^n)}
\]

where

\[
\Gamma_q (x) = 1 / \prod_{n=0}^\infty (1 - x q^n)
\]
Example: Resolved connifold

\[ Z_{\lambda, \mu^t} = \frac{\prod_{i<j}[\lambda_i - i - \lambda_j + j]q \prod_{i<j}[\mu_i - i - \mu_j + j]q}{\prod_{i,j}[\lambda_i - i - \mu_j + j + Q]q} \]

We write \( Q = a - a' \), and \( h_i = q^{a + \lambda_i - i + N} \), and \( h'_i = q^{a' + \mu_i - i + N} \),
\( \Delta(h) = \prod_{i<j}(h_i - h_j) \), \( \Delta(h, h') = \prod_{i,j}(h_i - h'_j) \):

\[ Z_{\lambda, \mu^t} \propto \frac{\Delta(h) \Delta(h')}{\Delta(h, h')} = \frac{\prod_{i<j}(h_i - h_j) \prod_{i<j}(h'_i - h'_j)}{\prod_{i,j}(h_i - h'_j)} \]

\[ = \det \left( \frac{1}{h_i - h'_j} \right) \quad \text{Cauchy determinant formula} \]

Rewrite:

\[ \det \left( \frac{1}{h_i - h'_j} \right) = \frac{1}{N!} \sum_{\sigma, \rho} (-1)^{\sigma \rho} \prod_{i} \frac{1}{h_{\rho(i)} - h'_i} \]
Transformation into a matrix integral

Laplace transform: \[ \frac{1}{h-h'} = \int_0^\infty dr \, e^{-r(h-h')} \]

\[
\det \left( \frac{1}{h_i - h'_j} \right) \\
= \frac{1}{N!} \sum_{\sigma, \rho} (-1)^{\sigma \rho} \prod_i \int_0^\infty dr_i e^{-r_i(h_{\rho(i)} - h'_{\sigma(i)})} \\
= \frac{1}{N!} \int_0^\infty dr_1 \ldots dr_N \det \left( e^{-r_i h_j} \right) \det \left( e^{r_i h'_j} \right) \\
\] Itzykson – Zuber matrix Integrals

\[ \frac{\Delta(h) \Delta(h')}{N!} \int_0^\infty dr_i \Delta(r)^2 \int_{U(N)^2} dU d\tilde{U} e^{-\text{tr} \, r(UhU^\dagger - \tilde{U}h'\tilde{U}^\dagger)} \]

\[ \frac{\Delta(h) \Delta(h')}{N!} \int_{H_N(\mathbb{R}^+)} dR \int_{U(N)^2} dU d\tilde{U} e^{-\text{tr} \, R(HUH^\dagger - \tilde{U}h'\tilde{U}^\dagger)} \]

Random Matrices and topological strings
Example: 2 boxes.
\[ Q_1 = a - a', \quad Q_2 = a' - a'' . \]

\[
Z = \sum_{\nu} \mathcal{Z}_{\lambda,a;\nu^t,a'} e^{-S|\nu|} \mathcal{Z}_{\nu,a';\mu^t,a''}.
\]

We write \( q = e^{-g_s} \), \( h_i(\nu) = q^{\nu_i - i + N + a'} \),
\( h = \text{diag}(h_i) \), \( \Delta(h) = \prod_{i<j}(h_i - h_j) \), \( H = UhU^\dagger \).

\[
Z = \sum_{\nu} \Delta(h(\nu))^2 e^{\frac{S}{g_s} \text{tr} \ln h(\nu)} \int_{H_N(\mathbb{R}^+)} dR \, dU \, e^{-\text{Tr} R(h(\lambda) - Uh(\nu)U^\dagger)} \int_{H_N(\mathbb{R}^+)} d\tilde{R} \, d\tilde{U} \, e^{-\text{Tr} \tilde{R}(\tilde{U}h(\nu)\tilde{U}^\dagger - h(\mu))} \int_{H_N(\mathbb{C})} \frac{dH}{\det f(q^{-a'}H)} \int_{H_N(\mathbb{R}^+)} dR \, d\tilde{R} \, e^{-\text{Tr} R(h(\lambda) - H)} e^{-\text{Tr} \tilde{R}(H - h(\mu))}.
\]

where \( f(x) = 1 / \Gamma_q(qx) \Gamma_q(1/x) \).
Matrix models and topological strings

Matrix model for the fiducial geometry, with one column of boxes:

\[
Z = \int dH_1 \ldots dH_{L-1} dR_1 \ldots dR_L e^{-\sum_i \text{Tr} R_i (H_{i-1} - H_i)} \\
\prod_i \frac{\text{det} (\Gamma_q (H_i / q^{a_i - 1})) \ \text{det} (\Gamma_q (H_i / q^{a_i + 1}))}{\text{det} (\Gamma_q (H_i / q^{a_i}))^2} \\
\prod_i e^{S_{i+1}^{a_i} g s \text{Tr} \ln H_i} \prod_i \frac{1}{\text{det} (f(q^{-a_i} H_i))} \\
\int dH_0 \frac{\text{det} (\Gamma_q (H_0 / q^{a_1}))}{\text{det} (\Gamma_q (H_0 / q^{a_0})) \ \prod_{i=1}^N \text{det} (H_0 - q^{\lambda_i - i + N + a_0})} \\
\int dH_L \frac{\text{det} (\Gamma_q (H_L / q^{a_{L-1}}))}{\text{det} (\Gamma_q (H_L / q^{a_{L}})) \ \prod_{i=1}^N \text{det} (H_L - q^{\mu_i - i + N + a_L})}
\]
Matrix models and topological strings

Matrix model for the fiducial geometry, with one column of boxes:

\[
Z = \int dH_1 \ldots dH_{L-1} dR_1 \ldots dR_L e^{-\sum_i \text{Tr} R_i (H_{i-1} - H_i)} \\
\prod_i \frac{\det (\Gamma_q (H_i / q^{a_i-1})) \det (\Gamma_q (H_i / q^{a_i+1}))}{\det (\Gamma_q (H_i / q^{a_i}))^2} \\
\prod_i e^{\frac{S_i + 1}{g_s} \text{Tr} \ln H_i} \prod_i \frac{1}{\det (f(q^{-a_i} H_i))} \\
\int dH_0 \frac{\det (\Gamma_q (H_0 / q^{a_1}))}{\det (\Gamma_q (H_0 / q^{a_0})) \prod_{i=1}^N \det (H_0 - q^{\lambda_i - i + N + a_0})} \\
\int dH_L \frac{\det (\Gamma_q (H_L / q^{a_{L-1}}))}{\det (\Gamma_q (H_L / q^{a_L})) \prod_{i=1}^N \det (H_L - q^{\mu_i - i + N + a_L})}
\]
Matrix model for the fiducial geometry, with several columns of boxes:

\[
Z = \int dH_1 \ldots dH_{L-1} dR_1 \ldots dR_L e^{-\sum_i \text{Tr} R_i (H_{i-1} - H_i)} \prod_{i,j} \frac{\det (\Gamma_q(H_i/q^{a_{i,j}})) \det (\Gamma_q(H_i/q^{a_{i+1,j}}))}{\det (\Gamma_q(H_i/q^{a_{i,j}}))^2} \prod_i e^{S_i + 1 \over g_s} \text{Tr} \ln H_i \prod_{i,j} \frac{1}{\det (f(q^{-a_{i,j}} H_i))}
\]
Matrix models and topological strings

Matrix model for the fiducial geometry, with several columns of boxes:

\[
Z = \int dH_1 \ldots dH_{L-1} \, dR_1 \ldots dR_L \, e^{-\sum_i \text{Tr} \, R_i (H_{i-1} - H_i)} \\
\prod_{i,j} \frac{\det (\Gamma_q (H_i / q^{a_{i-1,j}})) \det (\Gamma_q (H_i / q^{a_{i+1,j}}))}{\det (\Gamma_q (H_i / q^{a_{i,j}}))^2} \\
\prod_i e^{S_{i+1} \, \frac{\text{Tr} \, \ln H_i}{g_s}} \prod_{i,j} \frac{1}{\det (f(q^{-a_{i,j}} \, H_i))}
\]

Bertrand Eynard, IPHT CEA Saclay, CERN GENEZISS String Th
Matrix models and topological strings

Matrix model for the fiducial geometry, with several columns of boxes:

\[
Z = \int dH_1 \ldots dH_{L-1} \, dR_1 \ldots dR_L \, e^{-\sum_i \text{Tr} \, R_i (H_i - H_{i-1})} \prod_{i,j} \frac{\det (\Gamma_q (H_i / q^{a_{i,j}})) \, \det (\Gamma_q (H_i / q^{a_{i+1,j}}))}{\det (\Gamma_q (H_i / q^{a_{i,j}}))^2} \prod_i e^{\frac{S_i}{g_s} \text{Tr} \, \ln H_i} \prod_{i,j} \frac{1}{\det (f(q^{-a_{i,j}} H_i))}
\]
Matrix models and topological strings

Matrix model for the fiducial geometry, with several columns of boxes:

\[ Z = \int dH_1 \ldots dH_{L-1} \ dR_1 \ldots dR_L e^{-\sum_i \text{Tr} \ R_i (H_{i-1} - H_i)} \]

\[ \prod_{i,j} \frac{\det (\Gamma_q (H_i / q^{a_{i-1,j}})) \ \det (\Gamma_q (H_i / q^{a_{i+1,j}}))}{\det (\Gamma_q (H_i / q^{a_{i,j}}))^2} \]

\[ \prod_i e^{S_i^1 / g_s \text{Tr} \ \ln H_i} \prod_{i,j} \frac{1}{\det (f(q^{-a_{i,j}} H_i))} \]
Matrix models and topological strings

Matrix model for the fiducial geometry, with several columns of boxes:

\[
Z = \int dH_1 \ldots dH_{L-1} \, dR_1 \ldots dR_L \, e^{- \sum_i \text{Tr} R_i (H_{i-1} - H_i)} \left( \prod_{i,j} \frac{\det (\Gamma_q(H_i/q^{a_{i-1,j}})) \det (\Gamma_q(H_i/q^{a_{i+1,j}}))}{\det (\Gamma_q(H_i/q^{a_{i,j}}))^2} \right) \prod_i e^{\frac{S_{j+1}}{g_s} \text{Tr} \ln H_i} \prod_{i,j} \frac{1}{\det (f(q^{-a_{i,j}}H_i))}.
\]
Matrix models and topological strings

Matrix model for the fiducial geometry, with several columns of boxes:

\[
\begin{align*}
Z &= \int dH_1 \ldots dH_{L-1} dR_1 \ldots dR_L e^{-\sum_i \text{Tr} R_i (H_{i-1} - H_i)} \\
&\quad \times \prod_{i,j} \frac{\det (\Gamma_q (H_i / q^{a_{i-1,j}})) \det (\Gamma_q (H_i / q^{a_{i+1,j}}))}{\det (\Gamma_q (H_i / q^{a_{i,j}}))^2} \\
&\quad \times \prod_i e^{\frac{S_i + 1}{g_s} \text{Tr} \ln H_i} \prod_{i,j} \frac{1}{\det (f(q^{-a_{i,j}} H_i))}
\end{align*}
\]
Matrix model for the fiducial geometry, with several columns of boxes:

\[
Z = \int dH_1 \ldots dH_{L-1} dR_1 \ldots dR_L e^{-\sum_i \text{Tr} R_i (H_{i-1} - H_i)} \prod_{i,j} \frac{\det (\Gamma_q (H_i / q^{a_{i-1,j}})) \det (\Gamma_q (H_i / q^{a_{i+1,j}}))}{\det (\Gamma_q (H_i / q^{a_{i,j}}))^2} \prod_i e^{\frac{S_i+1}{g_s} \text{Tr} \ln H_i} \prod_{i,j} \frac{1}{\det (f(q^{-a_{i,j}} H_i))}
\]
Any CY3 toric geometry can be obtained from the box-fiducial geometry, by flop transitions.

Example: 4 boxes $\rightarrow$ local $\mathbb{P}^2$
Any CY3 toric geometry can be obtained from the box-fiducial geometry, by flop transitions.

Example: 4 boxes $\rightarrow$ local $\mathbb{P}^2$
Any CY3 toric geometry can be obtained from the box-fiducial geometry, by flop transitions.

**Example:** 4 boxes $\rightarrow$ local $\mathbb{P}^2$
Any CY3 toric geometry can be obtained from the box-fiducial geometry, by flop transitions.

**Example:** 4 boxes $\rightarrow$ local $\mathbb{P}^2$
Any CY3 toric geometry can be obtained from the box-fiducial geometry, by flop transitions.

**Example:** 4 boxes $\rightarrow$ local $\mathbb{P}^2$
Any CY3 toric geometry can be obtained from the box-fiducial geometry, by flop transitions.

Example: 4 boxes $\rightarrow$ local $\mathbb{P}^2$
Any CY3 toric geometry can be obtained from the box-fiducial geometry, by flop transitions.

Example: 4 boxes $\rightarrow$ local $\mathbb{P}^2$
Any CY3 toric geometry can be obtained from the box-fiducial geometry, by flop transitions.

Example: 4 boxes $\rightarrow$ local $\mathbb{P}^2$
Any CY3 toric geometry can be obtained from the box-fiducial geometry, by flop transitions.

Example: 4 boxes \(\rightarrow\) local \(\mathbb{P}^2\)
Any CY3 toric geometry can be obtained from the box-fiducial geometry, by flop transitions.

**Example:** 4 boxes $\rightarrow$ local $\mathbb{P}^2$
2. Matrix models
Consider the general chain of matrices:

\[ Z = \int dM_1 \ldots dM_L e^{-\frac{1}{g_s} \sum_i \text{Tr} V_i(M_i)} e^{\frac{1}{g_s} \sum_i c_i \text{Tr} M_i M_{i+1}} \]

Topological expansion:

\[ \ln Z = \sum_g g_s^{2g-2} F_g \]

**Question:** how to compute \( F_g \)'s?

**Method:** write and solve loop equations
Generalities about loop equations

General chain of matrices:

\[ Z = \int dM_1 \ldots dM_L e^{-\frac{1}{g_s} \sum_i \text{Tr} V_i(M_i)} e^{\frac{1}{g_s} \sum_i c_i \text{Tr} M_i M_{i+1}} \]

Loop equations = integration by parts = Schwinger-Dyson eqs. → give relationships among expectation values.

Example: \( \forall i, \quad \langle \text{Tr} V'_i(M_i) \rangle = c_i \langle \text{Tr} M_{i+1} \rangle + c_{i-1} \langle \text{Tr} M_{i-1} \rangle \)

Resolvents: \( W_i(x) = \langle \text{Tr} 1/(x - M_i) \rangle = \sum_g g_s^{2g-1} W_{1,i}^{(g)}(x) \) is such that:

- \( W_{1,i}^{(0)}(x) \) satisfies a non-linear equation,
- \( W_{1,i}^{(g)}(x) \) with \( g \geq 1 \) satisfy linear equations.
More generally: n-point correlation functions for $M_i$:

$$W_{n,i}(x_1, \ldots, x_n) = \left\langle \frac{1}{x_1 - M_i} \text{Tr} \frac{1}{x_2 - M_i} \cdots \text{Tr} \frac{1}{x_n - M_i} \right\rangle_c$$

$$= \sum_g g_s^{2g-2+n} W^{(g)}_{n,i}(x_1, \ldots, x_n)$$


For all chain of matrices: Loop equations imply that:

- $W^{(0)}_{1,i}(x)$ satisfies a non-linear equation = spectral curve of $M_i$,

- $W^{(g)}_{n,i}(x_1, \ldots, x_n)$ with $2g - 2 + n \geq 0$ satisfy a recursive set of linear equations. The solution of this recursion is the topological recursion.

- Then, knowing $W^{(g)}_{n,i}$ determines $F_g$: $\ln Z = \sum_g g_s^{2g-2} F_g$. 
The spectral curve \( y_i = W^{(0)}_{1,i}(x) \) can be found (for instance) from saddle point approximation.

The function \( x \mapsto y = W^{(0)}_{1,i}(x) \) defines a Riemann surface embedded into \( \mathbb{C} \times \mathbb{C} = \text{plane curve} \).

- **Bergman kernel** \( B(z_1, z_2) = \text{fundamental 2-form of the second kind on the spectral curve (derivative of the Green function)} \)

\[
W^{(0)}_{2,i}(x_1, x_2) dx_1 dx_2 = B(x_1, x_2) - \frac{dx_1 dx_2}{(x_1 - x_2)^2}
\]

- **Recursion kernel** \( K(z_0, z) = \frac{\int_{z'}=\bar{z} B(z_0, z')}{2(W^{(0)}_{1}(z) - W^{(0)}_{1}(\bar{z}))} dx(z) \)
The spectral curve \( y_i = W_{1,i}^{(0)}(x) \) can be found (for instance) from saddle point approximation.

The function \( x \mapsto y = W_{1,i}^{(0)}(x) \) defines a Riemann surface embedded into \( \mathbb{C} \times \mathbb{C} = \) plane curve.

- **Bergman kernel** \( B(z_1, z_2) = \) fundamental 2-form of the second kind on the spectral curve (derivative of the Green function)
  \[
  W_{2,i}^{(0)}(x_1, x_2)dx_1 dx_2 = B(x_1, x_2) - \frac{dx_1 dx_2}{(x_1 - x_2)^2}
  \]

- **Recursion kernel** \( K(z_0, z) = \)
  \[
  \frac{\int_{z' = \bar{z}}^z B(z_0, z')}{{2(W_1^{(0)}(z) - W_1^{(0)}(\bar{z}))}} dx(z)
  \]
The spectral curve $y_i = W_{1,i}^{(0)}(x)$ can be found (for instance) from saddle point approximation.

The function $x \mapsto y = W_{1,i}^{(0)}(x)$ defines a Riemann surface embedded into $\mathbb{C} \times \mathbb{C} = \text{plane curve}.$

- **Bergman kernel** $B(z_1, z_2) = \text{fundamental 2-form of the second kind on the spectral curve (derivative of the Green function)}$
  \[ W_{2,i}^{(0)}(x_1, x_2) dx_1 dx_2 = B(x_1, x_2) - \frac{dx_1 dx_2}{(x_1 - x_2)^2} \]

- **Recursion kernel** $K(z_0, z) = \frac{\int_{z'}^{z} B(z_0, z')}{2(W_1^{(0)}(z) - W_1^{(0)}(\bar{z}))} dx(z)$
When the spectral curve is known, then, all higher order correlation functions are found by the "topological recursion"

\[
W_{n+1}^{(g)}(x_0, J) = \sum_i \text{Res}_{z \to a_i} K(x_0, z) \left[ W_{n+2}^{(g-1)}(z, \bar{z}, J) \right]
\]

\[
\sum_h \sum_{I \subset J} W_{1+\#I}^{(h)}(z, I) W_{1+n-\#I}^{(g-h)}(\bar{z}, J \setminus I)
\]

Diagramatically \((J = \{x_1, \ldots, x_n\})\):
When the the $W_n^{(g)}$'s are known, then, the $F_g$'s are given by:

\[ \forall g \geq 2 \quad F_g = \frac{1}{2 - 2g} \sum_{i} \text{Res}_{x \to a_i} \Phi(x) W_1^{(g)}(x) \]

where $d\Phi/dx = y = W_1^{(0)}(x)$.

**Diagramatically:**

Property: $F_g = \text{symplectic invariant}$.

**Remark:** all spectral curves $y_i(x) = W_1^{(0)}(x)$ are symplectically equivalent. $\Rightarrow F_g(\{y_{i+1}(x)\}) = F_g(\{y_i(x)\})$.

$F_g$ is independent of which matrix $M_i$ we choose.
3. Remodeling the B-model
Remodeling the B-model

Partition function of topological strings on a toric CY3 $\mathcal{X}$:

$$\ln Z = \sum_g g_s^{2g-2} F_g$$

$F_g = \text{topological string amplitude of closed worldsheets of genus } g.$

$W_{n,i}^{(g)} = \text{topological string amplitude of "open" worldsheets of genus } g \text{ with } "n \text{ boundaries on a brane of type } i".$

Conjecture (Remodeling the B-model, Mariño 2006, BKMP 2008)

• The spectral curve $y = W_{1,i}^{(0)}(x)$ is (modulo symplectic transformation) the mirror curve of the target space $\mathcal{X}$.

• The higher $W_{n,i}^{(g)}$ and $F_g$ are obtained by the topological recursion.
Mirror symmetry: \( \text{CY } X \longrightarrow \text{CY } \hat{X} \)

If \( X \) is toric, then \( \hat{X} \) is an hyperbolic bundle over a complex plane curve:

\[
\hat{X} : \quad H(e^x, e^y) = uv
\]

where \( H = \text{polynomial} \).

Fibers degenerate over the mirror curve:

\[
H(e^x, e^y) = 0
\]
For the box geometry of size $n \times m$, mirror curve:

\[ H(e^x, e^y) = 0. \]

$H$ is a polynomial of degree $n, m$:

\[
H(X, Y) = \sum_{i=0}^{n} \sum_{j=0}^{m} H_{i,j} X^i Y^j
\]

Coefficients $H_{i,j}$ determined by periods

\[
\oint_{\mathcal{A}_{i,j}} y \, dx = \oint_{\mathcal{A}_{i,j}} \ln Y \frac{dX}{X} = t_{i,j} = \text{Kaehler moduli}
\]

Remark: # branchpoints = # vertices.
Mirror curve

For the box geometry of size \( n \times m \), mirror curve:
\( H(e^x, e^y) = 0. \)

\( H \) is a polynomial of degree \( n, m \):

\[
H(X, Y) = \sum_{i=0}^{n} \sum_{j=0}^{m} H_{i,j} X^j Y^i
\]

Coefficients \( H_{i,j} \) determined by periods

\[
\oint_{A_{i,j}} y \, dx = \oint_{A_{i,j}} \ln Y \, \frac{dX}{X} = t_{i,j} = \text{Kaehler moduli}
\]

Remark: \( \# \) branchpoints = \( \# \) vertices.
Mirror curve

For the box geometry of size \( n \times m \), mirror curve:
\[ H(e^x, e^y) = 0. \]
\( H \) is a polynomial of degree \( n, m \):
\[
H(X, Y) = \sum_{i=0}^{n} \sum_{j=0}^{m} H_{i,j} X^j Y^i
\]

Coefficients \( H_{i,j} \) determined by periods
\[
\int_{A_{i,j}} y \ dx = \int_{A_{i,j}} \ln Y \ \frac{dX}{X} = t_{i,j} = \text{Kaehler moduli}
\]

Remark: \# branchpoints = \# vertices.
For the box geometry of size $n \times m$, mirror curve:

$$H(e^x, e^y) = 0.$$ 

$H$ is a polynomial of degree $n, m$:

$$H(X, Y) = \sum_{i=0}^{n} \sum_{j=0}^{m} H_{i,j} X^j Y^i$$

Coefficients $H_{i,j}$ determined by periods

$$\int_{A_{i,j}} y \, dx = \int_{A_{i,j}} \ln Y \frac{dX}{X} = t_{i,j} = \text{Kaehler moduli}$$

Remark: $\# \text{ branchpoints} = \# \text{ vertices}$. 
Mirror curve

For the box geometry of size $n \times m$, mirror curve:

$$H(e^x, e^y) = 0.$$  

$H$ is a polynomial of degree $n, m$:

$$H(X, Y) = \sum_{i=0}^{n} \sum_{j=0}^{m} H_{i,j} X^i Y^j$$

Coefficients $H_{i,j}$ determined by periods

$$\oint_{A_{i,j}} y \, dx = \oint_{A_{i,j}} \ln Y \, \frac{dX}{X} = t_{i,j} = \text{Kaehler moduli}$$

Remark: $\# \text{ branchpoints} = \# \text{ vertices}$. 
4. Matrix model’s spectral curve
Towards a proof: The mirror curve satisfies 1), 2), 3)!

\[
Z = \int dM_1 \ldots dM_L \, e^{-\frac{1}{g_s} \sum_i \text{Tr} \, V_i(M_i)} \, e^{\frac{1}{g_s} \sum_i c_i \text{Tr} \, M_i M_{i+1}} \, M_i \in H_N(\prod_j \gamma_i^{n_{i,j}})
\]

Recipe: find a complex curve \( C \), and functions \( x_i(z) : C \rightarrow \mathbb{C}, \, i = 0, \ldots, L + 1 \), satisfying:

• 1) equations of motion: \( V_i'(x_i(z)) = c_{i-1} x_{i-1}(z) + c_i x_{i+1}(z) \).

• 2) filling fraction condition: \[
\frac{c_i}{2\pi i} \oint_{A_{i,j}} x_{i+1} \, dx_i = n_{i,j} / N g_s.
\]

• 3) endpoints: \( x_{i+1} \) has simple pole at \( x_i \) \( = \) endpoint for \( M_i \).

• 4) Find the choice which minimizes a certain action \( S_0 \).
Matrix model’s spectral curve

Towards a proof: **The mirror curve satisfies 1), 2), 3)**

\[
Z = \int dM_1 \ldots dM_L \ e^{-\frac{1}{g_s} \sum_i \text{Tr} V_i(M_i)} \ e^{\frac{1}{g_s} \sum_i c_i \text{Tr} M_i M_{i+1}}
\]

\[
M_i \in H_N(\prod_j \gamma_{i,j}^{n_{i,j}})
\]

Recipe: find a complex curve \( C \), and functions \( x_i(z) : C \to \mathbb{C}, i = 0, \ldots, L + 1 \), satisfying:

• 1) **equations of motion**: 
  
  \[
  V'_i(x_i(z)) = c_{i-1} x_{i-1}(z) + c_i x_{i+1}(z).
  \]

• 2) **filling fraction condition**: 
  
  \[
  \frac{c_i}{2\pi i} \int_{A_{i,j}} x_{i+1} \ dx_i = n_{i,j}/N g_s.
  \]

• 3) **endpoints**: \( x_{i+1} \) has simple pole at \( x_i = \text{endpoint for } M_i \).

• 4) Find the choice which **minimizes a certain action** \( S_0 \).
5. Conclusion
• We can construct an explicit matrix model for topological strings in any toric geometry.
• This gives the possibility to use all the toolbox of matrix models: integrability, orthogonal polynomials, loop equations, topological recursion, wall crossing (Riemann-Hilbert), etc...
• New light on mirror symmetry. The mirror curve and mirror map emerge from the matrix model.
• Non-perturbative definition of topological string. The matrix model is non-perturbative. All instantons resummations are known → background independence. Also → holomorphic anomaly equations [BCOV], integrability, quantum Riemann surfaces. Maybe link with AGT. Can we see other features of string theory in the matrix model? Dualities?