GR Assignments 01

1. Coordinate Transformations and Metrics

   Under a cooordinate transformation $\xi^A = \xi^A(x^\mu)$, the Minkowski metric η_{AB} transforms to a new metric $g_{\mu\nu}$ in such a way that proper distances are invariant. In other words, the line element $ds^2 = \eta_{AB} d\xi^A d\xi^B$ is invariant,

   $$\eta_{AB} d\xi^A d\xi^B = g_{\mu\nu} dx^\mu dx^\nu \quad . \tag{1}$$

   (a) Show that this implies that $g_{\mu\nu}$ is related to η_{AB} by

   $$g_{\mu\nu}(x) = \frac{\partial \xi^A}{\partial x^\mu} \frac{\partial \xi^B}{\partial x^\nu} \eta_{AB} \quad . \tag{2}$$

   (b) Determine the Minkowski metric $ds^2 = -dt^2 + dx^2$ in *Rindler coordinates* (T, X), $t = X \sinh T, x = X \cosh T$.

   (c) Show that the *inverse metric* $g^{\mu\nu}$, i.e. $g^{\mu\nu} g_{\nu\lambda} = \delta^\mu_{\ \lambda}$ is given by

   $$g^{\mu\nu} = \eta^{AB} \frac{\partial x^\mu}{\partial \xi^A} \frac{\partial x^\nu}{\partial \xi^B} \tag{3}$$

2. The Free Particle in Arbitrary Coordinates

   Transforming from inertial (Minkowski) coordinates ξ^A to arbitrary coordinates x^μ, the free-particle equation of motion $d^2\xi^A/d\tau^2 = 0$ becomes

   $$\frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu_{\ \nu\lambda} \frac{dx^\nu}{d\tau} \frac{dx^\lambda}{d\tau} = 0 \tag{4}$$

   where

   $$\Gamma^\mu_{\ \nu\lambda} = \frac{\partial x^\mu}{\partial \xi^A} \frac{\partial^2 \xi^A}{\partial x^\nu \partial x^\lambda} \quad . \tag{5}$$

   Show that this pseudo-force term Γ arising in the x^μ coordinate system is related to the metric $g_{\mu\nu}$ (eq. (2)) in these coordinates by

   $$\begin{aligned} \Gamma^\mu_{\ \nu\lambda} &= g^{\mu\rho} \Gamma_{\rho\nu\lambda} \\ \Gamma_{\mu\nu\lambda} &= \tfrac{1}{2}(g_{\mu\nu},_\lambda + g_{\mu\lambda},_\nu - g_{\nu\lambda},_\mu) \end{aligned} \tag{6}$$

   where $g^{\mu\nu}$ is the inverse metric and $g_{\mu\nu},_\lambda$ is short-hand for the partial derivative $(\partial g_{\mu\nu}/\partial x^\lambda)$.

3. Christoffel Symbols and Coordinate Transformations

The *Christoffel symbols* $\Gamma_{\mu\nu\lambda}$ and $\Gamma^{\mu}_{\ \nu\lambda}$ associated to a metric $g_{\mu\nu}$ are defined as in Exercise 2, eq. (6) (but now for an arbitrary metric). Manipulating and calculating these Christoffel symbols is one of the less pleasant tasks in GR - but it has got to be done. It is also a good warm-up exercise for learning how to manipulate (correctly!) objects with multiple indices, summations etc.

(a) Use the tensorial behaviour of the metric under a coordinate transformation $x^{\mu} \to y^{\mu'}(x^{\mu})$ to show that the Christoffel symbols transform as

$$\Gamma^{\mu'}_{\ \nu'\lambda'} = \Gamma^{\mu}_{\ \nu\lambda}\frac{\partial y^{\mu'}}{\partial x^{\mu}}\frac{\partial x^{\nu}}{\partial y^{\nu'}}\frac{\partial x^{\lambda}}{\partial y^{\lambda'}} + \frac{\partial y^{\mu'}}{\partial x^{\mu}}\frac{\partial^2 x^{\mu}}{\partial y^{\nu'}\partial y^{\lambda'}} \quad . \tag{7}$$

(b) Show that, as a consequence of (7), $\ddot{x}^{\mu} + \Gamma^{\mu}_{\ \nu\lambda}\dot{x}^{\nu}\dot{x}^{\lambda}$ transforms as a *vector* under coordinate transformations, i.e. that one has

$$\ddot{y}^{\mu'} + \Gamma^{\mu'}_{\ \nu'\lambda'}\dot{y}^{\nu'}\dot{y}^{\lambda'} = \frac{\partial y^{\mu'}}{\partial x^{\mu}}\left(\ddot{x}^{\mu} + \Gamma^{\mu}_{\ \nu\lambda}\dot{x}^{\nu}\dot{x}^{\lambda}\right) \tag{8}$$

Hint: you may have to use an identity which follows from differentiating

$$(\partial x^{\rho}/\partial y^{\mu'})(\partial y^{\mu'}/\partial x^{\lambda}) = \delta^{\rho}_{\ \lambda} \quad . \tag{9}$$

Remark: If you don't want to do 3a *and* 3b, then just do 3b, using the result (7) of 3a. First of all, it is easier (shorter), and secondly it is more rewarding / satisfactory to cancel non-tensorial terms rather than generate them …