1. Geodesics

(a) Geodesics and Euler-Lagrange Equations: Show that the Euler-Lagrange equations

\[
\frac{d}{d\tau} \frac{\partial L}{\partial \dot{x}^{\mu}} - \frac{\partial L}{\partial x^{\mu}} = 0 ,
\]  

for the Lagrangian

\[
L = \frac{1}{2} g_{\mu\nu} \frac{dx^{\mu}}{d\tau} \frac{dx^{\nu}}{d\tau}
\]

are the geodesic equations

\[
\frac{d^2 x^{\mu}}{d\tau^2} + \Gamma_{\mu\lambda}^{\nu} \frac{dx^{\nu}}{d\tau} \frac{dx^{\lambda}}{d\tau} = 0
\]

where, as usual,

\[
\Gamma_{\mu\nu\lambda}^{\rho} = g^{\mu\rho} \Gamma_{\nu\lambda}^{\rho} \\
\Gamma_{\mu\nu\lambda} = \frac{1}{2} (g_{\mu\nu,\lambda} + g_{\mu,\nu,\lambda} - g_{\nu,\lambda,\mu}) .
\]

(b) \( L \) is a Constant of Motion: Show that \( L \) is constant along any geodesic, i.e. that

\[
\frac{d}{d\tau} \left( g_{\mu\nu} \frac{dx^{\mu}}{d\tau} \frac{dx^{\nu}}{d\tau} \right) = 0
\]

for \( x^{\mu}(\tau) \) a solution of the geodesic equation.

(c) Geodesics on the Two-Sphere \( S^2 \): The metric of a 2-sphere with radius \( R \) is

\[
ds^2 = R^2 (d\theta^2 + \sin^2 \theta d\phi^2)
\]

Use (4) to calculate all its Christoffel symbols, show that the geodesic equations agree with the Euler-Lagrange equations of the Lagrangian

\[
L = \frac{1}{2} (\dot{\theta}^2 + \sin^2 \theta \dot{\phi}^2)
\]

and show that the great circles (longitudes) \( (\theta(\tau), \phi(\tau)) = (\tau, \phi_0) \) satisfy the geodesic equation.