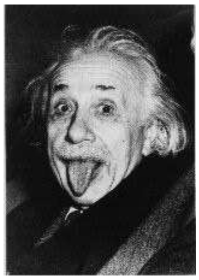

GR Assignments 02

1. Geodesics

   (a) Geodesics and Euler-Lagrange Equations: Show that the Euler-Lagrange equations
   $$\frac{d}{d\tau}\frac{\partial\mathcal{L}}{\partial\dot{x}^\mu}-\frac{\partial\mathcal{L}}{\partial x^\mu}=0\ , \tag{1}$$
   for the Lagrangian
   $$\mathcal{L}=\tfrac{1}{2}g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} \tag{2}$$
   are the geodesic equations
   $$\frac{d^2x^\mu}{d\tau^2}+\Gamma^\mu_{\ \nu\lambda}\frac{dx^\nu}{d\tau}\frac{dx^\lambda}{d\tau}=0\ , \tag{3}$$
   where, as usual,
   $$\begin{aligned}\Gamma^\mu_{\ \nu\lambda} &= g^{\mu\rho}\Gamma_{\rho\nu\lambda}\\ \Gamma_{\mu\nu\lambda} &= \tfrac{1}{2}(g_{\mu\nu},_\lambda+g_{\mu\lambda},_\nu-g_{\nu\lambda},_\mu)\ .\end{aligned} \tag{4}$$

   (b) $\mathcal{L}$ is a constant of motion: Show that $\mathcal{L}$ is constant along any geodesic, i.e. that
   $$\frac{d}{d\tau}\left(g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\right)=0 \tag{5}$$
   for $x^\mu(\tau)$ a solution of the geodesic equation.

   (c) Geodesics on the Two-Sphere S^2: The metric of a 2-sphere with radius R is
   $$ds^2=R^2(d\theta^2+\sin^2\theta d\phi^2)\ . \tag{6}$$
   Use (4) to calculate all its Christoffel symbols, show that the geodesic equations agree with the Euler-Lagrange equations of the Lagrangian
   $$\mathcal{L}=\tfrac{1}{2}(\dot{\theta}^2+\sin^2\theta\dot{\phi}^2)\ , \tag{7}$$
   and show that the great circles (longitudes) $(\theta(\tau),\phi(\tau))=(\tau,\phi_0)$ satisfy the geodesic equation.