1. Tensor Analysis I: Tensor Algebra

(a) Show that the partial derivative $\partial_\mu f(x) = \partial f(x)/\partial x^\mu$ of a scalar $f(x)$ transforms like (and hence is) a covector.

**Remark:** The partial derivative of a higher-rank tensor is not a tensor (you can easily check this yourself), forcing us to introduce a suitable tensorial generalisation of the partial derivative, the covariant derivative.

(b) Let $A_{\mu\nu}$ be a $(0, 2)$-tensor and $B^\mu$ a $(1, 0)$-tensor (a vector). Show that $A_{\mu\nu}B^\nu$ is a co-vector (i.e. transforms like a co-vector) and that $A_{\mu\nu}B^\mu B^\nu$ is a scalar.

**Remark:** In particular, given a metric $g_{\mu\nu}$, the scalar product $g_{\mu\nu}V^\mu W^\nu$ of two vectors $V$ and $W$ is indeed a scalar in the tensorial sense.

(c) Let $V^\mu(x)$ be a vector field and denote by $\partial_\mu = \partial/\partial x^\mu$ the partial derivatives. Show that the first-order linear differential operator

$$V(x) = V^\mu(x) \partial_\mu$$

is invariant under coordinate transformations. Analogously, let $A_\mu(x)$ be a covector. Show that

$$A(x) = A_\mu(x)dx^\mu$$

is invariant under coordinate transformations.

**Remark:** It is extremely useful to think of vector fields in this way. The basic coordinate-independent object is $V$. $V$ can be expanded in a basis $\partial_\mu$, and its components with respect to this basis are the $V^\mu$. If you change coordinates, the basis changes, and therefore also the components of $V$ change when expanded with respect to this new basis.

2. The Effective Geodesic Potential

Generalising the discussion in section 24.1 of the lecture notes, and following Remark 4 at the end of that section, derive the effective potential equation for the general class of static spherically symmetric metrics of the form

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2d\Omega^2$$

$$f(r) = 1 + 2\phi(r)$$

(this includes e.g. possibly electrically and / or magnetically charged stars and black holes, and / or in the presence of a cosmological constant).