1. Tensor Analysis II: the Covariant Derivative

The covariant derivative $\nabla_\mu$ is the tensorial generalisation of the partial derivative $\partial_\mu$, i.e. it is such that the covariant derivative of a tensor is again a tensor. More precisely, the covariant derivative of a $(p, q)$-tensor is then a $(p, q + 1)$-tensor because it has one more lower (covariant) index. Since the partial derivative $\partial_\mu f$ of a scalar $f$ (a $(0, 0)$-tensor) is a covector (a $(0, 1)$-tensor), one sets $\nabla_\mu f = \partial_\mu f$. However, as we have seen, the partial derivative $\partial_\mu V^\nu$ of a vector $V^\nu$ (a $(1, 0)$-tensor) is *not* a $(1, 1)$-tensor. This can be rectified by defining the covariant derivative of a vector to be

$$\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu_{\mu\lambda} V^\lambda . \quad (1)$$

It can be checked that this is indeed a tensor, the non-tensorial nature of the partial derivative cancelling exactly against that of the Christoffel symbols. A similar story holds for covectors: the partial derivative $\partial_\mu A^\nu$ of a $(0, 1)$-tensor (covector) is *not* a tensor. This can be cured in the same way as for vectors, and one can check that

$$\nabla_\mu A^\nu = \partial_\mu A^\nu - \Gamma^{\lambda}_{\mu\nu} A^\lambda \quad (2)$$

is indeed a $(0, 2)$-tensor. The action of $\nabla_\mu$ on vectors and covectors can be extended to arbitrary $(p, q)$-tensors. For instance, for a $(0, 2)$-tensor $B_{\mu\nu}$ one has

$$\nabla_\lambda B_{\mu\nu} = \partial_\lambda B_{\mu\nu} - \Gamma^\rho_{\lambda\mu} B_{\rho\nu} - \Gamma^\rho_{\lambda\nu} B_{\rho\mu} . \quad (3)$$

(a) An alternative way to arrive at (2) is to demand the Leibniz rule for the covariant derivative of a product of tensors: deduce (2) from (1) using the fact that $A_\nu V^\nu$ is a scalar for any vector $V^\nu$, so that $\nabla_\mu (A_\nu V^\nu) = \partial_\mu (A_\nu V^\nu)$, and using the Leibniz rule for $\partial_\mu$ (i.e. $\partial_\mu (A_\nu V^\nu) = (\partial_\mu A_\nu) V^\nu + A_\nu \partial_\mu V^\nu$) and $\nabla_\mu$.

(b) Check that, even though $\partial_\mu A_\nu$ is *not* a tensor, the curl (or rotation) $\partial_\mu A_\nu - \partial_\nu A_\mu$ is (i.e. transforms as) a tensor. Then show that the covariant curl of a covector is equal to its ordinary curl,

$$\nabla_\mu A_\nu - \nabla_\nu A_\mu = \partial_\mu A_\nu - \partial_\nu A_\mu . \quad (4)$$
This provides an alternative argument for the fact that $\partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu}$ is a tensor.

(c) Show that (3) implies that the covariant derivative of the metric is zero, $\nabla_\lambda g_{\mu\nu} = 0$.

2. Radial Fall & the Repulsive Reissner-Nordstrøm Core

The Reissner-Nordstrøm metric

$$ds^2 = -(1 - \frac{2m}{r} + \frac{q^2}{r^2})dt^2 + (1 - \frac{2m}{r} + \frac{q^2}{r^2})^{-1}dr^2 + r^2d\Omega^2$$

(5)
is a solution to the coupled Einstein-Maxwell equations describing the gravitational field of a spherically symmetric electrically charged star ($m \sim$ mass, $q \sim$ charge). We will assume $m^2 > q^2$.

You have already determined the effective potential for this metric in exercise 03.2. Now consider the special case of radially infalling (angular momentum $L = 0$) massive particles.

(a) Show that for $q \neq 0$ radially infalling (and electrically neutral) massive particles cannot reach $r = 0$ and are reflected by the Reissner-Nordstrøm metric (there is thus a repulsive gravitational force (anti-gravity . . . ) at the core of the Reissner-Nordstrøm solution) and compare and contrast this with the behaviour of radially infalling particles in the Schwarzchild geometry.

(b) Determine the turning point $r_{\text{min}}$ of the trajectory of a radially infalling massive particle starting out at rest at infinity (this gives a condition on $E$ or $E_{\text{eff}}$) and show that $r_{\text{min}} < r_-$, where $r_-$ is the smaller of the two roots of the equation $f(r) = 1 - 2m/r + q^2/r^2 = 0$. 

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