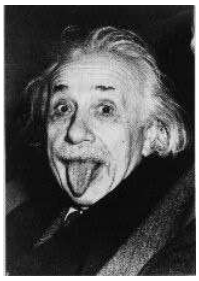

GR Assignments 04 (Supplement)

A Useful Identity Involving the Determinant of the Metric

Let M be a real symmetric matrix, and denote by $\det M$ its determinant. The result that we want to establish is that, under a variation of M, $M \to M + \delta M$, the determinant changes as

$$\delta \det M = (\det M)\,\mathrm{tr}(M^{-1}\delta M) \quad . \tag{1}$$

In particular, choosing M to be a metric tensor $g_{\mu\nu}$, and defining $g := |\det g_{\mu\nu}|$, one has the result

$$g^{-1}\delta g = g^{\mu\nu}\delta g_{\mu\nu} \tag{2}$$

One proof of this result, based on the explicit expansion of a determinant in terms of minors,

$$g = \sum_{\nu} (-1)^{\mu+\nu} g_{\mu\nu} |m_{\mu\nu}| \quad , \tag{3}$$

is described in the lecture notes (section 4.6). Here is another proof (it is phrased as an exercise, but you don't have to do this).

1. We begin with the following general and very useful formula for the determinant of a (symmetric) matrix M,

$$\det M = \exp \mathrm{tr} \log M \quad . \tag{4}$$

   Establish this result.

   Hint: First estabish this result for diagonal matrices D. Then use the fact that any symmetric matrix can be diagonalised by an orthogonal transformation (rotation) R, $M = R^T D R = R^{-1} D R$.

2. Now we need to learn how to vary (or differentiate) functions of matrices. Thus let $f(x)$ be an ordinary function, defined by a Taylor expansion around $x = 0$, say (we could also consider the case $x = 1$ etc.),

$$f(x) = \sum_n f_n \frac{x^n}{n!} \quad . \tag{5}$$

   Then the function $f(M)$ is defined by

$$f(M) = \sum_n f_n \frac{M^n}{n!} \quad . \tag{6}$$

Now we want to vary (differentiate) this function with respect to x or M, i.e. we want to consider the effect of replacing x by $x + \delta x$ or M by $M + \delta M$. For $f(x)$ one obviously finds the result

$$\delta f(x) = f'(x)\delta x \tag{7}$$

where $f'(x)$ has the Taylor expansion

$$f'(x) = \sum_n f_n \frac{x^{n-1}}{(n-1)!} \quad . \tag{8}$$

If one varies $f(M)$ in the same way, one does not find a simple result because M and δM need not commute, e.g. $\delta(M^2) = (\delta M)M + M\delta M \neq 2M\delta M$. To avoid this problem, we will switch from matrix-valued functions of matrices to real-valued functions of matrices by considering, instead of $f(M)$, its trace

$$F(M) := \operatorname{tr} f(M) \quad . \tag{9}$$

Since one can cyclically commute matrices in the trace, you should now find

$$\delta \operatorname{tr} f(M) = \operatorname{tr}(f'(M)\delta M) \quad . \tag{10}$$

The important message here is that traces of functions of matrices behave exactly like ordinary functions.

3. Now let us consider specifically the logarithm. $\log x$ and x^{-1} have Taylor expansions around 1, $x = 1 + y$, given by

$$\begin{aligned} \log(1+y) &= \sum_{n=0}^{\infty} (-1)^n \frac{y^{n+1}}{(n+1)} \\ (1+y)^{-1} &= \sum_{n=0}^{\infty} (-1)^n y^n \quad . \end{aligned} \tag{11}$$

Varying $\log x$, one then finds the familiar result

$$\delta \log x = x^{-1}\delta x \quad , \tag{12}$$

where we have identified $\delta x \equiv \delta y$.

For matrices, you should now show that, analogously

$$\delta \operatorname{tr} \log M = \operatorname{tr} M^{-1}\delta M \quad , \tag{13}$$

where $\log M$ and M^{-1} are defined via their Taylor expansion around the identity matrix $\mathbb{I}$, $M = \mathbb{I} + A$, with $\delta M = \delta A$.

Remark: By continuing to use, as in step (1), the diagonalisability of M, one can replace steps (2) and (3) by the following, perhaps more elementary argument.

We define a variation δM of M in the following way: let $M(\alpha)$ be a family (curve) of matrices with $M(0) = M$, and define the variation δM of M to be the tangent vector to the curve $M(\alpha)$ at $\alpha = 0$,

$$\delta M = \frac{d}{d\alpha} M(\alpha)|_{\alpha=0} \tag{14}$$

Then introduce a family of matrices $R(\alpha)$ which diagonalise $M(\alpha)$, $M(\alpha) = R(\alpha)^T D(\alpha) R(\alpha)$. Thus the variation of $\det M$ is

$$\delta \operatorname{tr} \log M = \frac{d}{d\alpha} \left(\operatorname{tr} \log M(\alpha)\right)_{\alpha=0} = \frac{d}{d\alpha} \left(\operatorname{tr} \log D(\alpha)\right)_{\alpha=0} \quad . \tag{15}$$

Since now all the matrices are diaognal, one finds

$$\delta \operatorname{tr} \log M = \operatorname{tr} D(0)^{-1} D'(0) \quad . \tag{16}$$

Re-expressing this in terms of M, and using

$$R^T(\alpha) R(\alpha) = \mathbb{I} \quad \forall \ \alpha \quad \Rightarrow \quad R'^T(0) R(0) + R^T(0) R'(0) = 0 \tag{17}$$

one can now deduce the identity (13).

4. We have now assembled all the bits and pieces we need to study the variation of $\det M$. Using the fundamental identity (4) and the results established above, show that the variation of the determinant can be written as in (1),

$$\delta \det M = (\det M) \operatorname{tr}(M^{-1} \delta M) \quad . \tag{18}$$

5. Denote by $g := |\det(g_{\mu\nu})|$ the absolute value of the determinant of the metric. Then the above identity (18) becomes

$$\delta g = g g^{\mu\nu} \delta g_{\mu\nu} \quad . \tag{19}$$

Show that this can also be written as

$$\delta g = -g g_{\mu\nu} \delta g^{\mu\nu} \quad . \tag{20}$$

6. Show that (19) implies that the partial derivatives of g are related to the partial derivatives of the metric by

$$\partial_\lambda g = g g^{\mu\nu} g_{\mu\nu,\lambda} \quad . \tag{21}$$