1. Stationary and Freely Falling Schwarzschild Observers

(a) Consider a stationary observer (sitting at fixed values of \((r > 2m, \theta, \phi)\)) in the Schwarzschild geometry

\[
ds^2 = -\left(1 - \frac{2m}{r}\right)dt^2 + \left(1 - \frac{2m}{r}\right)^{-1}dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2). \tag{1}
\]

Determine his worldline 4-velocity \(u^\alpha = dx^\alpha/d\tau\) and the acceleration \(a^\alpha = \nabla_\tau u^\alpha \equiv u^\beta \nabla_\beta u^\alpha\) and calculate \(g_{\alpha\beta}a^\alpha a^\beta\). What happens as \(r \to \infty\) and \(r \to 2m\)?

(b) Consider a freely (and radially) falling observer in the Schwarzschild geometry, initially at rest at radius \(r(\tau = 0) \equiv R > 2m\). Show that the proper time it would (formally) take him to reach \(r = 0\) is (up to factors of \(c\)) given by

\[
\tau = \pi \left(\frac{R^3}{8m}\right)^{1/2}. \tag{2}
\]

Estimate this for \(R\) the radius of the sun \((R \sim 7 \times 10^{10} \text{ cm})\) and \(2m\) its Schwarzschild radius \((2m \sim 3 \times 10^5 \text{ cm})\), restoring the correct factors of \(c\), and show that this is of the order of an hour.

Remark: this can be interpreted as an estimate for the time of complete collapse of a star under its own gravitational attraction.

2. Kruskal Coordinates for the Schwarzschild Space-Time

To get to a completely non-singular and complete form of the Schwarzschild metric introduce the so-called Kruskal coordinates

\[
X = \frac{1}{2}(e^{(t + r^*/4m}) + e^{-(t - r^*)/4m}),
\]

\[
T = \frac{1}{2}(e^{(t + r^*)/4m}) - e^{-(t - r^*)/4m}. \tag{3}
\]

where the tortoise coordinate \(r^*\) is defined by \(r^* = r + 2m \log(r/2m - 1)\).

Show that in terms of these coordinates the metric is

\[
ds^2 = \frac{32m^3}{r}e^{-r/2m}(dT^2 + dX^2) + r^2d\Omega^2, \tag{4}
\]

where \(r = r(T, X)\) is implicitly given by

\[
X^2 - T^2 = (r/2m - 1)e^{r/2m}. \tag{5}
\]