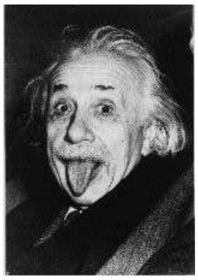

# GR Assignments 04

1. Stationary and Freely Falling Schwarzschild Observers

   (a) Consider a stationary observer (sitting at fixed values of $(r > 2m, \theta, \phi)$) in the Schwarzschild geometry

   $$ds^2 = -\left(1 - \frac{2m}{r}\right) dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \quad . \tag{1}$$

   Determine his worldline 4-velocity $u^\alpha = dx^\alpha/d\tau$ and the acceleration $a^\alpha = \nabla_\tau u^\alpha \equiv u^\beta \nabla_\beta u^\alpha$ and calculate $g_{\alpha\beta} a^\alpha a^\beta$. What happens as $r \to \infty$ and $r \to 2m$?

   (b) Consider a freely (and radially) falling observer in the Schwarzschild geometry, initially at rest at radius $r(\tau = 0) \equiv R > 2m$. Show that the proper time it would (formally) take him to reach $r = 0$ is (up to factors of $c$) given by

   $$\tau = \pi \left(\frac{R^3}{8m}\right)^{1/2} \quad . \tag{2}$$

   Estimate this for $R$ the radius of the sun ($R \sim 7 \times 10^{10}$ cm) and $2m$ its Schwarzschild radius ($2m \sim 3 \times 10^5$ cm), restoring the correct factors of $c$, and show that this is of the order of an hour.

   **Remark:** this can be interpreted as an estimate for the time of complete collapse of a star under its own gravitational attraction.

2. Kruskal Coordinates for the Schwarzschild Space-Time

   To get to a completely non-singular and complete form of the Schwarzschild metric introduce the so-called *Kruskal coordinates*

   $$\begin{aligned} X &= \tfrac{1}{2}(\mathrm{e}^{(t + r^*)/4m} + \mathrm{e}^{-(t - r^*)/4m}) \\ T &= \tfrac{1}{2}(\mathrm{e}^{(t + r^*)/4m} - \mathrm{e}^{-(t - r^*)/4m}) \quad . \end{aligned} \tag{3}$$

   where the *tortoise coordinate* $r^*$ is defined by $r^* = r + 2m \log(r/2m - 1)$.

   Show that in terms of these coordinates the metric is

   $$ds^2 = \frac{32m^3}{r} \mathrm{e}^{-r/2m}(-dT^2 + dX^2) + r^2 d\Omega^2 \quad , \tag{4}$$

   where $r = r(T, X)$ is implicitly given by

   $$X^2 - T^2 = (r/2m - 1)\mathrm{e}^{r/2m} \quad . \tag{5}$$