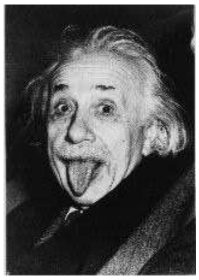

GR Assignments 05

1. Tensor Analysis II: the Covariant Derivative

   The *covariant derivative* ∇_μ is the tensorial generalisation of the partial derivative ∂_μ, i.e. it is such that the covariant derivative of a tensor is again a tensor. More precisely, the covariant derivative of a (p,q)-tensor is then a $(p,q+1)$-tensor because it has one more lower (covariant) index. Since the partial derivative $\partial_\mu f$ of a scalar f (a $(0,0)$-tensor) is a covector (a $(0,1)$-tensor), one sets $\nabla_\mu f = \partial_\mu f$. However, as we have seen, the partial derivative $\partial_\mu V^\nu$ of a vector V^ν (a $(1,0)$-tensor) is *not* a $(1,1)$-tensor. This can be rectified by defining the covariant derivative of a vector to be

   $$\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu_{\mu\lambda} V^\lambda \quad . \tag{1}$$

   It can be checked that this is indeed a tensor, the non-tensorial nature of the partial derivative of a vector cancelling exactly against that of the Christoffel symbols. A smilar story holds for covectors: the partial derivative $\partial_\mu A_\nu$ of a $(0,1)$-tensor (covector) is *not* a tensor. This can be cured in the same way as for vectors, and one can check that

   $$\nabla_\mu A_\nu = \partial_\mu A_\nu - \Gamma^\lambda_{\mu\nu} A_\lambda \tag{2}$$

   is indeed a $(0,2)$-tensor. The action of ∇_μ on vectors and covectors can be extended to arbitrary (p,q)-tensors. For instance, for a $(0,2)$-tensor $B_{\mu\nu}$ one has

   $$\nabla_\lambda B_{\mu\nu} = \partial_\lambda B_{\mu\nu} - \Gamma^\rho{}_{\lambda\mu} B_{\rho\nu} - \Gamma^\rho{}_{\lambda\nu} B_{\mu\rho} \quad . \tag{3}$$

   (a) An alternative way to arrive at (2) is to demand the *Leibniz rule* for the covariant derivative of a product of tensors: deduce (2) from (1) using the fact that $A_\nu V^\nu$ is a scalar for any vector V^ν, so that $\nabla_\mu(A_\nu V^\nu) = \partial_\mu(A_\nu V^\nu)$ and using the Leibniz rule for ∂_μ (i.e. $\partial_\mu(A_\nu V^\nu) = (\partial_\mu A_\nu)V^\nu + A_\nu \partial_\mu V^\nu$) and ∇_μ.

   (b) Check that, even though $\partial_\mu A_\nu$ is *not* a tensor, the *curl* (or *rotation*) $\partial_\mu A_\nu - \partial_\nu A_\mu$ *is* (i.e. transforms as) a tensor. Then show that the covariant curl of a covector is equal to its ordinary curl,

   $$\nabla_\mu A_\nu - \nabla_\nu A_\mu = \partial_\mu A_\nu - \partial_\nu A_\mu \quad . \tag{4}$$

This provides an alternative argument for the fact that $\partial_\mu A_\nu - \partial_\nu A_\mu$ is a tensor.

(c) Show that (3) implies that the covariant derivative of the metric is zero, $\nabla_\lambda g_{\mu\nu} = 0$.

2. TENSOR ANALYSIS III: THE COVARIANT DIVERGENCE AND THE LAPLACIAN

An extremely useful identity for the variation (in particular, the derivative) of the determinant $g := |\det g_{\mu\nu}|$ of the metric is

$$g^{-1}\delta g = g^{\mu\nu}\delta g_{\mu\nu} \qquad\qquad g^{-1}\partial_\lambda g = g^{\mu\nu}\partial_\lambda g_{\mu\nu} \quad . \tag{5}$$

One proof of this result, based on the explicit expansion of a determinant in terms of minors, $g = \sum_\nu (-1)^{\mu+\nu} g_{\mu\nu}|m_{\mu\nu}|$ is described in the lecture notes (section 4.5). Another easy proof can be deduced from the fundamental matrix identity $\det M = \exp \operatorname{tr} \log M$.

(a) Use (5) to show that the contracted Christoffel symbol $\Gamma^\mu_{\ \mu\lambda}$ (summation over the index μ) can be calculated by the simple formula

$$\Gamma^\mu_{\ \mu\lambda} = g^{-1/2}\partial_\lambda g^{+1/2} \quad . \tag{6}$$

(b) Show that this implies that the covariant divergence of a vector (current) J^μ and an anti-symmetric tensor $F^{\mu\nu} = -F^{\nu\mu}$ can be written as

$$\nabla_\mu J^\mu = g^{-1/2}\partial_\mu(g^{1/2}J^\mu) \qquad\qquad \nabla_\mu F^{\mu\nu} = g^{-1/2}\partial_\mu(g^{1/2}F^{\mu\nu}) \quad . \tag{7}$$

(c) Use the formula

$$\Box\Phi = g^{-1/2}\partial_\mu(g^{1/2}g^{\mu\nu}\partial_\nu\Phi) \tag{8}$$

to determine the Laplacian in $\mathbb{R}^3$ in spherical coordinates (r, θ, ϕ).