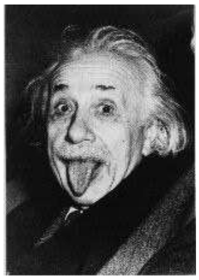

GR Assignments 06

1. Properties of the Riemann Curvature Tensor

In the course, we defined the Riemann curvature tensor via the commutator of covariant derivatives,

$$[\nabla_\mu, \nabla_\nu]V^\lambda = R^\lambda_{\ \sigma\mu\nu}V^\sigma \quad . \tag{1}$$

and we defined its contractions, the Ricci tensor $R_{\alpha\beta} = R^\gamma_{\ \alpha\gamma\beta}$ and the Ricci scalar $R = g^{\alpha\beta}R_{\alpha\beta}$. The Riemann curvature tensor has the symmetries

$$\begin{aligned} &(I) \quad R_{\alpha\beta\gamma\delta} = -R_{\alpha\beta\delta\gamma} \qquad (II) \quad R_{\alpha\beta\gamma\delta} = -R_{\beta\alpha\gamma\delta} \\ &(III) \quad R_{\alpha[\beta\gamma\delta]} = 0 \ \Leftrightarrow \ R_{\alpha\beta\gamma\delta} + R_{\alpha\gamma\delta\beta} + R_{\alpha\delta\beta\gamma} = 0 \end{aligned} \tag{2}$$

[see section 6.3 of the lecture notes for proofs and make sure that you understand the details!]

(a) Show that the above symmetries imply the property

$$(IV) \quad R_{\alpha\beta\gamma\delta} = R_{\gamma\delta\alpha\beta} \quad . \tag{3}$$

(b) Show that symmetry (IV) implies that the Ricci tensor is symmetric.

(c) Like any linear operator, the covariant derivative ∇_α satisfies the Jacobi identity

$$[\nabla_\alpha, [\nabla_\beta, \nabla_\gamma]] + \text{cyclic permutations} = 0 \quad . \tag{4}$$

[If you like, you can check this explicitly by writing out all the commutators.] Show that this, together with the definition (1), implies the *Bianchi identity*

$$\nabla_\alpha R_{\mu\nu\beta\gamma} + \text{cyclic permutations in } (\alpha,\beta,\gamma) = 0 \tag{5}$$

(d) By contracting the Bianchi identity over the indices (μ,β) (multiplication by $g^{\mu\beta}$) and (ν,α) (multiplication by $g^{\nu\alpha}$), deduce the identity (*contracted Bianchi identity*)

$$\nabla_\alpha(2R^\alpha_{\ \gamma} - \delta^\alpha_{\ \gamma}R) = 0 \tag{6}$$

and show that this is equivalent to the statement that the *Einstein tensor* $G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R$ has vanishing covariant divergence, $\nabla^\alpha G_{\alpha\beta} = 0$.

Remark: In all these equations indices are lowered and raised with the metric and its inverse: $R_{\alpha\beta\gamma\delta} = g_{\alpha\lambda}R^\lambda_{\ \beta\gamma\delta}$, $\nabla^\alpha = g^{\alpha\rho}\nabla_\rho$, $R^\alpha_{\ \gamma} = g^{\alpha\beta}R_{\beta\gamma}$ etc.

2. Curvature in 2 Dimensions

In 2 dimensions, there is only one independent component of the curvature tensor, say R_{1212} (this one component is equivalent to the *Gauss curvature* of a 2-dimensional surface). As a consequence, there must be a simple relation between R_{1212} and the scalar curvature.

(a) Using the definition $R = g^{\alpha\beta}R_{\alpha\beta} = g^{\alpha\beta}R^{\lambda}{}_{\alpha\lambda\beta}$, the anti-symmetry of the Riemann tensor in its first and second pair of indices, and the fact that in two dimensions the inverse metric is explicitly given by

$$\left(g^{\alpha\beta}\right) = \frac{1}{g_{11}g_{22} - g_{12}g_{21}} \begin{pmatrix} g_{22} & -g_{12} \\ -g_{21} & g_{11} \end{pmatrix} \tag{7}$$

show that the Riemann tensor and the Ricci scalar are related by

$$R = \frac{2}{g_{11}g_{22} - g_{12}g_{21}} R_{1212} \quad . \tag{8}$$

(b) Calculate the scalar curvature of the metric $ds^2 = dx^2 + e^{2x}dy^2$ on $\mathbb{R}^2$.
Remark: You should find that the scalar curvature is constant and *negative*! Note that this metric is some sort of exponential or hyperbolic analogue of the (trigonometric) metric $dx^2 + \sin^2 x\, dy^2$ on the sphere. Calculate the Christoffel symbols from the geodesic equation, then determine one component of the Riemann tensor, e.g. R_{xyxy}.

3. The Geodesic Deviation Equation (section 6.4)

(a) Consider two geodesics $x^\mu(\tau)$ and $x^\mu(\tau) + \delta x^\mu(\tau)$,

$$\begin{aligned} &\tfrac{d^2}{d\tau^2}x^\mu + \Gamma^\mu_{\nu\lambda}(x)\tfrac{d}{d\tau}x^\nu \tfrac{d}{d\tau}x^\lambda = 0 \quad , \\ &\tfrac{d^2}{d\tau^2}(x^\mu + \delta x^\mu) + \Gamma^\mu_{\nu\lambda}(x+\delta x)\tfrac{d}{d\tau}(x^\nu + \delta x^\nu)\tfrac{d}{d\tau}(x^\lambda + \delta x^\lambda) = 0 \quad . \end{aligned} \tag{9}$$

with $\delta x^\mu(\tau)$ an infinitesimal *deviation vector.* Show that this implies the evolution equation

$$\tfrac{d^2}{d\tau^2}\delta x^\mu + 2\Gamma^\mu_{\nu\lambda}(x)\tfrac{d}{d\tau}x^\nu \tfrac{d}{d\tau}\delta x^\lambda + \partial_\rho \Gamma^\mu_{\nu\lambda}(x)\delta x^\rho \tfrac{d}{d\tau}x^\nu \tfrac{d}{d\tau}x^\lambda = 0 \tag{10}$$

for the deviation vector $\delta x^\mu(\tau)$.

(b) Show that (10) can be written in manifestly covariant form as

$$(D_\tau)^2(\delta x^\mu) = R^\mu{}_{\nu\lambda\rho}\dot{x}^\nu \dot{x}^\lambda \delta x^\rho \quad . \tag{11}$$

Hint: It is easier to start from (11) and to deduce (10). Also: you will of course have to use the fact that $x^\mu(\tau)$ itself satisfies the geodesic equation.