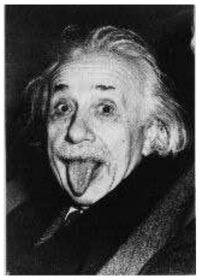

GR Assignments 01

1. Coordinate Transformations and Metrics in Minkowski Space

   Under a cooordinate transformation $\xi^a = \xi^a(x^\mu)$, the Minkowski metric η_{ab} transforms to a new metric $g_{\mu\nu}$ in such a way that proper distances are invariant,

   $$\eta_{ab} d\xi^a d\xi^b = g_{\mu\nu} dx^\mu dx^\nu \quad \Rightarrow \quad g_{\mu\nu} = J^a_\mu J^b_\nu \eta_{ab} \tag{1}$$

   with $J^a_\mu = \partial\xi^a / \partial x^\mu$ the Jacobi matrix of the transformation.

   (a) Determine the components of the (1+1)-dimensional Minkowski metric (equivalently the line element $ds^2 = -dt^2 + dx^2$) in *Rindler coordinates* (T, X), $t = X \sinh T, x = X \cosh T$.

   (b) Show that the *inverse metric* $g^{\mu\nu}$, is given by

   $$g^{\mu\nu} = \eta^{ab} \frac{\partial x^\mu}{\partial \xi^a} \frac{\partial x^\nu}{\partial \xi^b} \equiv \eta^{ab} J^\mu_a J^\nu_b \tag{2}$$

   i.e. show that $g^{\mu\nu} g_{\nu\lambda} = \delta^\mu_\lambda$ (make sure in this calculation that any index appears at most twice, and that only if that index is to be summed over!).

2. Free Relativistic Particle in Arbitrary Coordinates

   As you saw in the lecture, transforming from inertial (Minkowski) coordinates ξ^a to arbitrary coordinates x^μ, the free-particle equation of motion $d^2\xi^a / d\tau^2 = 0$ becomes

   $$\frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\lambda} \frac{dx^\nu}{d\tau} \frac{dx^\lambda}{d\tau} = 0 \tag{3}$$

   where

   $$\Gamma^\mu_{\nu\lambda} = \frac{\partial x^\mu}{\partial \xi^a} \frac{\partial^2 \xi^a}{\partial x^\nu \partial x^\lambda} \equiv J^\mu_a J^a_{\nu\lambda} \ . \tag{4}$$

   Show that this pseudo-force term Γ arising in the x^μ coordinate system is related to the metric $g_{\mu\nu}$ (eq. (1)) in these coordinates by

   $$\begin{aligned} \Gamma^\mu_{\nu\lambda} &= g^{\mu\rho} \Gamma_{\rho\nu\lambda} \\ \Gamma_{\mu\nu\lambda} &= \tfrac{1}{2}(g_{\mu\nu},_\lambda + g_{\mu\lambda},_\nu - g_{\nu\lambda},_\mu) \end{aligned} \tag{5}$$

   where $g^{\mu\nu}$ is the inverse metric and $g_{\mu\nu},_\lambda$ is short-hand for the partial derivative $(\partial g_{\mu\nu} / \partial x^\lambda)$.

3. GEODESICS

(a) GEODESICS AND EULER-LAGRANGE EQUATIONS: Show that the Euler-Lagrange equations

$$\frac{d}{d\tau}\frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} - \frac{\partial \mathcal{L}}{\partial x^\mu} = 0 \ , \tag{6}$$

for the Lagrangian

$$\mathcal{L} = \tfrac{1}{2} g_{\mu\nu} \frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} \tag{7}$$

are the geodesic equations

$$\frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu_{\ \nu\lambda}\frac{dx^\nu}{d\tau}\frac{dx^\lambda}{d\tau} = 0 \ , \tag{8}$$

where the *Christoffel symbols* $\Gamma^\mu_{\ \nu\lambda}$ associated to a metric $g_{\mu\nu}$ are defined as in Exercise 2 (eq. 5), but now for an arbitrary metric.

(b) $\mathcal{L}$ IS A CONSTANT OF MOTION: Show that $\mathcal{L}$ is constant along any geodesic, i.e. that

$$\frac{d}{d\tau}\left(g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\right) = 0 \tag{9}$$

for $x^\mu(\tau)$ a solution of the geodesic equation.

(c) GEODESICS ON THE TWO-SPHERE S^2: The metric of a 2-sphere with radius R is

$$ds^2 = R^2(d\theta^2 + \sin^2\theta d\phi^2) \ . \tag{10}$$

Calculate all its Christoffel symbols, show that the geodesic equations agree with the Euler-Lagrange equations of the Lagrangian

$$\mathcal{L} = \tfrac{1}{2}(\dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2) \ , \tag{11}$$

and show that the great circles (longitudes) $(\theta(\tau), \phi(\tau)) = (\tau, \phi_0)$ satisfy the geodesic equation.