GR Assignments 02 (Optional)

Christoffel Symbols and Coordinate Transformations

One of the exercises I usually give is to determine the (non-tensorial) transformation behaviour of the Christoffel symbols $\Gamma_{\mu\nu\lambda}$ and $\Gamma^{\mu}_{\ \nu\lambda}$ associated to a metric $g_{\mu\nu}$, defined as

$$\begin{aligned}\Gamma^{\mu}_{\ \nu\lambda} &= g^{\mu\rho}\Gamma_{\rho\nu\lambda} \\ \Gamma_{\mu\nu\lambda} &= \tfrac{1}{2}(g_{\mu\nu},_{\lambda} + g_{\mu\lambda},_{\nu} - g_{\nu\lambda},_{\mu}) \ ,\end{aligned} \tag{1}$$

under a general coordinate transformation $x^\mu \to y^\alpha$, and to show that as a consequence of this non-tensoriality the geodesic equation

$$\ddot{x}^\mu + \Gamma^{\mu}_{\ \nu\lambda}\dot{x}^\nu \dot{x}^\lambda = 0 \tag{2}$$

does transform nicely (as a vector, with the Jacobi matrix) under such transformations.

This is a bit tedious and not a lot of fun, but it is a good check on your understanding of the formalism and your ability to manipulate correctly and in an accident-free manner objects with multiple indices and summations etc. Therefore, if you are interested in seriously learning and working with general relativity, I recommend that you try to do this exercise yourself at some point (but I will provide you with the solution in case you get stuck). So here is the exercise:

1. Use the tensorial behaviour of the metric under a coordinate transformation $x^\mu \to y^\alpha(x^\mu)$,
$$g_{\alpha\beta} = J^\mu_\alpha J^\nu_\beta g_{\mu\nu} \tag{3}$$
to show that the Christoffel symbols transform as ($J^\mu_{\beta\gamma} = \partial_\beta J^\mu_\gamma = \partial^2 x^\mu / \partial y^\beta \partial y^\gamma$)
$$\Gamma^{\alpha}_{\ \beta\gamma} = J^\alpha_\mu J^\nu_\beta J^\lambda_\gamma \Gamma^{\mu}_{\ \nu\lambda} + J^\alpha_\mu J^\mu_{\beta\gamma} \tag{4}$$

2. Show that, as a consequence of (4), $\ddot{x}^\mu + \Gamma^{\mu}_{\ \nu\lambda}\dot{x}^\nu\dot{x}^\lambda$ transforms as a *vector* under coordinate transformations, i.e. that one has
$$\ddot{y}^\alpha + \Gamma^{\alpha}_{\ \beta\gamma}\dot{y}^\beta\dot{y}^\gamma = J^\alpha_\mu \left(\ddot{x}^\mu + \Gamma^{\mu}_{\ \nu\lambda}\dot{x}^\nu\dot{x}^\lambda\right) \tag{5}$$
Hint: you may have to use an identity which follows from differentiating
$$J^\alpha_\mu J^\mu_\beta = \delta^\alpha_\beta \ \ . \tag{6}$$

SOLUTION (OUTLINE):

1. For partial derivatives one has the chain rule $\partial_\gamma = J^\lambda_\gamma \partial_\lambda$ ("∂_λ is a covector"). Therefore for the partial derivatives of the metric one has

$$g_{\alpha\beta,\gamma} = (J^\mu_\alpha J^\nu_\beta g_{\mu\nu})_{,\gamma} = g_{\mu\nu,\lambda} J^\mu_\alpha J^\nu_\beta J^\lambda_\gamma + (J^\mu_{\alpha\gamma} J^\nu_\beta + J^\mu_\alpha J^\nu_{\beta\gamma}) g_{\mu\nu} \quad . \tag{7}$$

Adding up the 3 terms comprising the Christoffel symbol $\Gamma_{\alpha\beta\gamma}$, one obtains

$$\begin{aligned} 2\Gamma_{\alpha\beta\gamma} &= g_{\alpha\beta,\gamma} + g_{\alpha\gamma,\beta} - g_{\beta\gamma,\alpha} \\ &= 2 J^\mu_\alpha J^\nu_\beta J^\lambda_\gamma \Gamma_{\mu\nu\lambda} \\ &\quad + (J^\mu_{\alpha\gamma} J^\nu_\beta + J^\mu_\alpha J^\nu_{\beta\gamma} + J^\mu_{\alpha\beta} J^\nu_\gamma + J^\mu_\alpha J^\nu_{\gamma\beta} - J^\mu_{\beta\alpha} J^\nu_\gamma - J^\mu_\beta J^\nu_{\gamma\alpha}) g_{\mu\nu} \quad . \end{aligned} \tag{8}$$

In the last line, the 3rd term cancels against the 5th (because $J^\mu_{\alpha\beta}$ is symmetric), the 1st term cancels against the 6th (because $J^\mu_{\alpha\gamma}$ and $g_{\mu\nu}$ are symmetric), while the 2nd and 4th term add up, so that one finds

$$\Gamma_{\alpha\beta\gamma} = J^\mu_\alpha J^\nu_\beta J^\lambda_\gamma \Gamma_{\mu\nu\lambda} + J^\mu_\alpha J^\nu_{\beta\gamma} g_{\mu\nu} \quad . \tag{9}$$

Now the hard work has been done. Raising the 1st index of the Christoffel symbol, using the inverse metric

$$g^{\alpha\delta} = g^{\sigma\rho} J^\alpha_\sigma J^\delta_\rho \quad , \tag{10}$$

it is now simple to see that one obtains the claimed result (4),

$$\Gamma^\alpha_{\ \beta\gamma} = g^{\alpha\delta} \Gamma_{\delta\beta\gamma} = J^\alpha_\mu J^\nu_\beta J^\lambda_\gamma \Gamma^\mu_{\ \nu\lambda} + J^\alpha_\mu J^\mu_{\beta\gamma} \quad . \tag{11}$$

For example, for the 2nd term one has (just using properties of inverse Jacobi matrices and metrics)

$$\begin{aligned} g^{\alpha\delta} J^\mu_\delta J^\nu_{\beta\gamma} g_{\mu\nu} &= g^{\sigma\rho} J^\alpha_\sigma J^\delta_\rho J^\mu_\delta J^\nu_{\beta\gamma} g_{\mu\nu} = g^{\sigma\rho} J^\alpha_\sigma \delta^\mu_\rho J^\nu_{\beta\gamma} g_{\mu\nu} \\ &= g^{\sigma\mu} J^\alpha_\sigma J^\nu_{\beta\gamma} g_{\mu\nu} = \delta^\sigma_\nu J^\alpha_\sigma J^\nu_{\beta\gamma} = J^\alpha_\nu J^\nu_{\beta\gamma} \end{aligned} \tag{12}$$

2. The 4-velocities transform as vectors (the chain rule again), $\dot{y}^\alpha = J^\alpha_\mu \dot{x}^\mu$. Therefore for the acceleration one has

$$\ddot{y}^\alpha = J^\alpha_\mu \ddot{x}^\mu + J^\alpha_{\mu\nu} \dot{x}^\mu \dot{x}^\nu \quad . \tag{13}$$

Therefore

$$\begin{aligned} \ddot{y}^\alpha + \Gamma^\alpha_{\ \beta\gamma} \dot{y}^\beta \dot{y}^\gamma &= J^\alpha_\mu (\ddot{x}^\mu + \Gamma^\mu_{\ \nu\lambda} J^\nu_\beta J^\lambda_\gamma \dot{y}^\beta \dot{y}^\gamma) + J^\alpha_\mu J^\mu_{\beta\gamma} \dot{y}^\beta \dot{y}^\gamma + J^\alpha_{\mu\nu} \dot{x}^\mu \dot{x}^\nu \\ &= J^\alpha_\mu (\ddot{x}^\mu + \Gamma^\mu_{\ \nu\lambda} \dot{x}^\nu \dot{x}^\lambda) + (J^\alpha_\mu J^\mu_{\beta\gamma} + J^\alpha_{\mu\nu} J^\mu_\beta J^\nu_\gamma) \dot{y}^\beta \dot{y}^\gamma \end{aligned} \tag{14}$$

The 1st term will give us the desired result, and cooperatively the 2nd term is identically zero because (use $\partial_\gamma = J^\nu_\gamma \partial_\nu$ again)

$$0 = (\delta^\alpha_\beta)_{,\gamma} = (J^\alpha_\mu J^\mu_\beta)_{,\gamma} = J^\alpha_{\mu\nu} J^\nu_\gamma J^\mu_\beta + J^\alpha_\mu J^\mu_{\beta\gamma} \quad . \tag{15}$$