GR Assignments 02 (Optional)

Christoffel Symbols and Coordinate Transformations

One of the exercises I usually give is to determine the (non-tensorial) transformation behaviour of the Christoffel symbols $\Gamma_{\mu\nu\lambda}$ and $\Gamma^\mu_{\nu\lambda}$ associated to a metric $g_{\mu\nu}$, defined as

$$\Gamma_{\nu\lambda} = g^{\rho\nu} \Gamma_{\rho\nu\lambda},$$
$$\Gamma_{\mu\nu\lambda} = \frac{1}{2} (g_{\mu\nu,\lambda} + g_{\mu\lambda,\nu} - g_{\nu\lambda,\mu}),$$

under a general coordinate transformation $x^\mu \rightarrow y^\alpha$, and to show that as a consequence of this non-tensoriality the geodesic equation

$$\ddot{x}^\mu + \Gamma^\mu_{\nu\lambda} \dot{x}^\nu \dot{x}^\lambda = 0$$

does transform nicely (as a vector, with the Jacobi matrix) under such transformations.

This is a bit tedious and not a lot of fun, but it is a good check on your understanding of the formalism and your ability to manipulate correctly and in an accident-free manner objects with multiple indices and summations etc. Therefore, if you are interested in seriously learning and working with general relativity, I recommend that you try to do this exercise yourself at some point (but I will provide you with the solution in case you get stuck). So here is the exercise:

1. Use the tensorial behaviour of the metric under a coordinate transformation $x^\mu \rightarrow y^\alpha(x^\mu)$,

$$g_{\alpha\beta} = J^\mu_\alpha J^\mu_\beta g_{\mu\nu}$$

(3)

to show that the Christoffel symbols transform as $(J^\mu_\beta J^\nu_\gamma - \partial_\beta J^\nu_\gamma = \partial^2 x^\mu / \partial y^\beta \partial y^\gamma)$

$$\Gamma^\alpha_{\beta\gamma} = J^\alpha_\mu J^\nu_\beta J^\lambda_\gamma \Gamma^\mu_{\nu\lambda} + J^\alpha_\mu J^\mu_\beta J^\lambda_\gamma$$

(4)

2. Show that, as a consequence of (4), $\ddot{x}^\mu + \Gamma^\mu_{\nu\lambda} \dot{x}^\nu \dot{x}^\lambda$, transforms as a vector under coordinate transformations, i.e. that one has

$$\ddot{y}^\alpha + \Gamma^\alpha_{\beta\gamma} \dot{y}^\beta \dot{y}^\gamma = J^\alpha_\mu \left( \ddot{x}^\mu + \Gamma^\mu_{\nu\lambda} \dot{x}^\nu \dot{x}^\lambda \right)$$

(5)

Hint: you may have to use an identity which follows from differentiating

$$J^\alpha_\mu J^\mu_\beta = \delta^\alpha_\beta.$$  

(6)
Solution (Outline):

1. For partial derivatives one has the chain rule \( \partial_\gamma = J_\lambda^\gamma \partial_\lambda \) ("\( \partial_\lambda \) is a covector"). Therefore for the partial derivatives of the metric one has

\[
g_{\alpha \beta, \gamma} = (J^\mu_{\alpha \gamma} J^\nu_{\beta \gamma} g_{\mu \nu})_{,\gamma} = g_{\mu \nu, \lambda} J^\mu_{\alpha \gamma} J^\nu_{\beta \gamma} + (J^\mu_{\alpha \gamma} J^\nu_{\beta \gamma} + J^\mu_{\alpha \gamma} J^\nu_{\beta \gamma}) g_{\mu \nu} \ .
\]  

Adding up the 3 terms comprising the Christoffel symbol \( \Gamma_{\alpha \beta \gamma} \), one obtains

\[
2 \Gamma_{\alpha \beta \gamma} = g_{\alpha \beta, \gamma} + g_{\alpha \gamma, \beta} - g_{\beta \gamma, \alpha}
= 2 J^\mu_{\alpha \gamma} J^\nu_{\beta \gamma} \Gamma_{\nu \mu \lambda} + (J^\mu_{\alpha \gamma} J^\nu_{\beta \gamma} + J^\mu_{\alpha \gamma} J^\nu_{\beta \gamma} - J^\mu_{\beta \alpha} J^\nu_{\gamma \gamma} - J^\mu_{\beta \alpha} J^\nu_{\gamma \gamma}) g_{\mu \nu} \ .
\]  

In the last line, the 3rd term cancels against the 5th (because \( J^\mu_{\alpha \beta} \) is symmetric), the 1st term cancels against the 6th (because \( J^\mu_{\alpha \gamma} \) and \( g_{\mu \nu} \) are symmetric), while the 2nd and 4th term add up, so that one finds

\[
\Gamma_{\alpha \beta \gamma} = J^\mu_{\alpha \gamma} J^\nu_{\beta \gamma} \Gamma_{\nu \mu \lambda} + J^\mu_{\alpha \gamma} J^\nu_{\beta \gamma} g_{\mu \nu} \ .
\]  

Now the hard work has been done. Raising the 1st index of the Christoffel symbol, using the inverse metric

\[
g^{\alpha \delta} = g^{\sigma \rho} J_\sigma^\alpha J_\rho^\delta \ ,
\]

it is now simple to see that one obtains the claimed result (4),

\[
\Gamma_{\beta \gamma} = g^{\alpha \delta} \Gamma_{\delta \beta \gamma} = J^\alpha_{\beta \gamma} J^\nu_{\gamma \nu} \Gamma_{\nu \mu \lambda} + J^\alpha_{\beta \gamma} J^\nu_{\gamma \nu} g_{\mu \nu} \ .
\]  

For example, for the 2nd term one has (just using properties of inverse Jacobi matrices and metrics)

\[
g^{\alpha \delta} J^\mu_{\delta \gamma} g_{\mu \nu} = g^{\sigma \rho} J_\sigma^\alpha J_\rho^\delta J^\nu_{\delta \gamma} g_{\mu \nu} = g^{\sigma \rho} J_\sigma^\alpha \delta_{\rho \gamma} g_{\mu \nu} = g^{\sigma \rho} J_\sigma^\alpha J^\nu_{\delta \gamma} g_{\mu \nu} = g^{\sigma \rho} J_\sigma^\alpha J^\nu_{\beta \gamma} = J^\nu_{\beta \gamma} \Gamma_{\nu \mu \lambda} + J^\nu_{\beta \gamma} J^\nu_{\gamma \nu} g_{\mu \nu} \ .
\]  

2. The 4-velocities transform as vectors (the chain rule again), \( \dot{y}^\alpha = J^\alpha_{\mu \nu} \dot{x}^\mu \). Therefore for the acceleration one has

\[
\ddot{y}^\alpha = J^\alpha_{\mu \nu} \ddot{x}^\mu + J^\alpha_{\mu \nu} \dot{x}^\mu \dot{x}^\nu .
\]  

Therefore

\[
\ddot{y}^\alpha + \Gamma^\alpha_{\beta \gamma} \dot{y}^\beta \dot{y}^\gamma = J^\alpha_{\mu \nu} (\ddot{x}^\mu + \Gamma^\mu_{\nu \lambda} J^\nu_{\beta \gamma} \dot{x}^\lambda \dot{y}^\gamma) + J^\alpha_{\mu \nu} \dot{x}^\mu \dot{x}^\nu + J^\alpha_{\mu \nu} \dot{x}^\mu \dot{x}^\nu
= J^\alpha_{\mu \nu} (\ddot{x}^\mu + \Gamma^\mu_{\nu \lambda} \dot{x}^\lambda \dot{x}^\mu) + (J^\alpha_{\mu \nu} J^\nu_{\beta \gamma} + J^\alpha_{\mu \nu} J^\nu_{\beta \gamma}) \ddot{y}^\beta \dot{y}^\gamma .
\]  

The 1st term will give us the desired result, and cooperatively the 2nd term is identically zero because (use \( \partial_\gamma = J^\nu_{\beta \gamma} \partial_\nu \) again)

\[
0 = (\delta^\beta_\nu)_{,\gamma} = (J^\alpha_{\mu \nu} J^\nu_{\beta \gamma})_{,\gamma} = J^\alpha_{\mu \nu} J^\nu_{\beta \gamma} J^\nu_{\gamma \delta} + J^\alpha_{\mu \nu} J^\nu_{\beta \gamma} .
\]