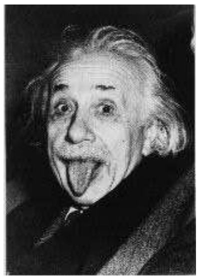

GR Assignments 05

1. Kruskal Coordinates for the Schwarzschild Space-Time

   To get to a maximal non-singular form of the Schwarzschild metric (for any $r \neq 0$), introduce the so-called *Kruskal coordinates*

$$\begin{aligned} X &= \tfrac{1}{2}(\mathrm{e}^{(t+r^*)/4m} + \mathrm{e}^{-(t-r^*)/4m}) \\ T &= \tfrac{1}{2}(\mathrm{e}^{(t+r^*)/4m} - \mathrm{e}^{-(t-r^*)/4m}) \ . \end{aligned} \tag{1}$$

   where the *tortoise coordinate* r^* is defined by $r^* = r + 2m\log(r/2m - 1)$.

   (a) Show that in terms of these coordinates the metric is

$$ds^2 = \frac{32m^3}{r}\mathrm{e}^{-r/2m}(-dT^2 + dX^2) + r^2 d\Omega^2 \ , \tag{2}$$

   where $r = r(T,X)$ is implicitly given by

$$X^2 - T^2 = (r/2m - 1)\mathrm{e}^{r/2m} \ . \tag{3}$$

   (b) Express the locations of the Schwarzschild radius ($r = 2m$) and the central singularity ($r = 0$) in terms of X and T.

2. Painlevé-Gullstrand Coordinates for the Schwarzschild Space-Time

   In the Schwarzschild coordinates (t,r), the Schwarzschild metric has the standard form

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2 d\Omega^2 \qquad f(r) = 1 - \frac{2m}{r} \ . \tag{4}$$

   (a) Show that the metric

$$ds^2 = -f(r)dT^2 + 2C(r)dTdr + f(r)^{-1}(1 - C(r)^2)dr^2 + r^2 d\Omega^2 \tag{5}$$

   is equivalent to the Schwarzschild metric for *any* function $C(r)$. [**Hint:** Begin with (4) and consider the coordinate transformation $T(t,r) = t + \psi(r)$.]

   (b) Now choose $C(r)$ such that $g_{rr} = 1$ (Painlevé-Gullstrand (PG) coordinates). Write down the resulting metric and show that it is completely non-singular for all $r > 0$ (in particular for $r \to 2m$), i.e. show that the metric coefficients are bounded and the determinant is non-zero.

(c) Show that the choice $C(r) = 1$ gives rise to the metric in Eddington-Finkelstein coordinates (with $T \equiv v = t + r^*$).

Optional Further Exercises:

Test your understanding/knowledge of GR

(solutions will *not* be provided - see the Lecture notes for details).

The metric in PG coordinates is related to timelike geodesics in the same way as the metric in Eddington-Finkelstein coordinates is related to null geodesics. To see this, consider the field of normal vectors $u_\alpha = -\partial_\alpha T$ orthogonal to the surfaces of constant T (in Scharzschild coordinates $x^\alpha = (t, r, \ldots)$).

(d) Show that $u^\alpha u_\alpha = -1$. Then show that in general the two properties $u^\alpha u_\alpha =$ const and $u_\alpha = -\partial_\alpha T$ imply that u^α is geodesic, i.e. $u^\beta \nabla_\beta u^\alpha = 0$.

(e) Show that the geodesics $x^\alpha(\tau)$ to which the u^α are tangent ($u^\alpha = \dot{x}^\alpha$) are radial geodesics ($L = 0$) with proper time $\tau = T$ and energy $E = 1$ (corresponding to observers that would have started off at rest at $r = \infty$).