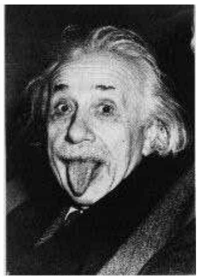

GR Assignments 06

1. Properties of the Riemann Curvature Tensor

   In the course, we defined the Riemann curvature tensor via the commutator of covariant derivatives,
   $$[\nabla_\mu, \nabla_\nu]V^\lambda = R^\lambda_{\ \sigma\mu\nu}V^\sigma \quad . \tag{1}$$
   and we defined its contractions, the Ricci tensor $R_{\alpha\beta} = R^\gamma_{\ \alpha\gamma\beta}$ and the Ricci scalar $R = g^{\alpha\beta}R_{\alpha\beta}$. The Riemann curvature tensor has the symmetries
   $$R_{\alpha\beta\gamma\delta} = -R_{\alpha\beta\delta\gamma} = -R_{\beta\alpha\gamma\delta} \quad , \quad R_{\alpha[\beta\gamma\delta]} = 0 \Leftrightarrow R_{\alpha\beta\gamma\delta} + R_{\alpha\gamma\delta\beta} + R_{\alpha\delta\beta\gamma} = 0 \tag{2}$$
   [see section 7.3 of the lecture notes for proofs and make sure that you understand the details!]

   (a) Show that the above symmetries imply that the Ricci tensor is symmetric.

   (b) Like any linear operator, the covariant derivative ∇_α satisfies the Jacobi identity
   $$[\nabla_\alpha, [\nabla_\beta, \nabla_\gamma]] + \text{cyclic permutations} = 0 \quad . \tag{3}$$
   Show that this, together with the definition (1), implies the *Bianchi identity*
   $$\nabla_\alpha R_{\mu\nu\beta\gamma} + \text{cyclic permutations in } (\alpha, \beta, \gamma) = 0 \tag{4}$$

   (c) By double-contraction of the Bianchi identity, deduce the *contracted Bianchi identity*
   $$\nabla_\alpha(2R^\alpha_{\ \gamma} - \delta^\alpha_{\ \gamma}R) = 0 \tag{5}$$
   and show that this is equivalent to the statement that the *Einstein tensor* $G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R$ has vanishing covariant divergence, $\nabla^\alpha G_{\alpha\beta} = 0$.

   Remark: In all these equations indices are lowered and raised with the metric and its inverse: $R_{\alpha\beta\gamma\delta} = g_{\alpha\lambda}R^\lambda_{\ \beta\gamma\delta}$, $\nabla^\alpha = g^{\alpha\rho}\nabla_\rho$, $R^\alpha_{\ \gamma} = g^{\alpha\beta}R_{\beta\gamma}$ etc.

2. On the Klein-Gordon Field in a Curved Space-Time

   The action of a real (free, massice) scalar field ϕ in a gravitational background $g_{\alpha\beta}$ is
   $$S[\phi, g_{\alpha\beta}] = \int \sqrt{g}d^4x\ L \equiv -\tfrac{1}{2}\int \sqrt{g}d^4x \left(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + m^2\phi^2\right) \tag{6}$$

The resulting equation of motion is $\left(\Box - m^2\right)\phi = 0$ (make sure that you know how to derive this!), and the generally covariant energy-momentum tensor is

$$T_{\alpha\beta} = \partial_\alpha\phi\partial_\beta\phi + g_{\alpha\beta}L \tag{7}$$

(a) Show that $T_{\alpha\beta}$ is (covariantly) conserved when ϕ is a solution to the Klein-Gordon equation of motion, i.e.

$$\left(\Box - m^2\right)\phi = 0 \quad \Rightarrow \quad \nabla^\alpha T_{\alpha\beta} = 0 \ . \tag{8}$$

(b) Show that $T_{\alpha\beta}$ is related to the variation of the action *with respect to the metric* by

$$\delta S = -\tfrac{1}{2}\int \sqrt{g}d^4x\ T_{\alpha\beta}\delta g^{\alpha\beta} \tag{9}$$

3. On the Maxwell Equations in a Curved Space-Time

The Maxwell action in a gravitational background is

$$S[A_\alpha, g_{\alpha\beta}] = \int \sqrt{g}d^4x\ L = -\tfrac{1}{4}\int \sqrt{g}d^4x F_{\alpha\beta}F^{\alpha\beta} \tag{10}$$

The resulting vacuum Maxwell equations are $\nabla_\alpha F^{\alpha\beta} = 0$ (make sure that you know how to derive this!), and the gauge-invariant and generally covariant energy momentum tensor is

$$T_{\alpha\beta} = F_{\alpha\gamma}F_\beta^{\ \gamma} - \tfrac{1}{4}g_{\alpha\beta}F_{\gamma\delta}F^{\gamma\delta} \tag{11}$$

(a) Use the Maxwell equations

$$\nabla_\alpha F^{\alpha\beta} = 0 \quad , \quad \nabla_\alpha F_{\beta\gamma} + \text{cyclic permutations} = 0 \tag{12}$$

to deduce the covariant conservation law

$$\nabla^\alpha T_{\alpha\beta} = 0 \ . \tag{13}$$

Remark: This is a tensorial equation. For such calculations you should just use the properties of the covariant derivative and **not** write out the covariant derivative in terms of the non-tensorial Christoffel symbols and partial derivatives.

Hint: Instead of embarking blindly on this calculation, remind yourself first how to do the calculation in Minkowski space. Exactly the same procedure should then work in general. If done correctly, this should be a four-line calculation.

(b) Show that the energy momentum tensor is related to the variation of the Maxwell action with respect to the metric in the same way as for the scalar field in (9).

Hint: don't forget the implicit metric-dependence in expressions like $F_{\alpha\beta}F^{\alpha\beta}$.