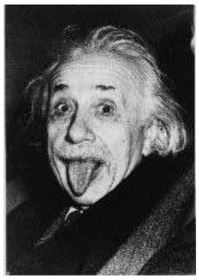

GR Assignments 03

1. Tensor Analysis I: Tensor Algebra

   Let $V^\mu(x)$ be a vector field and denote by $\partial_\mu = \partial/\partial x^\mu$ the partial derivatives. Show that the first-order linear differential operator

   $$V(x) = V^\mu(x)\,\partial_\mu \tag{1}$$

   is invariant under coordinate transformations. Analogously, let $A_\mu(x)$ be a covector. Show that

   $$A(x) = A_\mu(x)dx^\mu \tag{2}$$

   is invariant under coordinate transformations.

   Remark: It is extremely useful to think of vector fields in this way. The basic *coordinate-independent* object is V. V can be expanded in a basis ∂_μ, and its components with respect to this basis are the V^μ. If you change coordinates, the basis changes, and therefore also the components of V change when expanded with respect to this new basis.

2. Tensor Analysis II: the Covariant Derivative

   The covariant derivative of a covector field A_μ is

   $$\nabla_\mu A_\nu = \partial_\mu A_\nu - \Gamma^\lambda_{\mu\nu} A_\lambda \tag{3}$$

   Show that, even though $\partial_\mu A_\nu$ is *not* a tensor (which is why the Γ-term is required in $\nabla_\mu A_\nu$), the *curl* (or *rotation*) $\partial_\mu A_\nu - \partial_\nu A_\mu$ *is* (i.e. transforms as) a tensor. Then show that the covariant curl of a covector is equal to its ordinary curl,

   $$\nabla_\mu A_\nu - \nabla_\nu A_\mu = \partial_\mu A_\nu - \partial_\nu A_\mu \ . \tag{4}$$

   This provides an alternative argument for the fact that $\partial_\mu A_\nu - \partial_\nu A_\mu$ is a tensor.

3. The Effective Geodesic Potential

   Generalising the discussion in section 24.1 of the lecture notes, and following Remark 5 at the end of that section, derive the effective potential equation for the general class of static spherically symmetric metrics of the form

   $$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2 d\Omega^2 \quad , \quad f(r) = 1 + 2\phi(r) \tag{5}$$

   (this includes e.g. possibly electrically and / or magnetically charged stars and black holes, and / or in the presence of a cosmological constant).