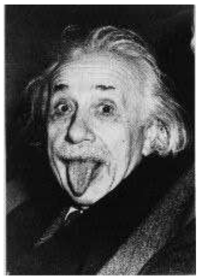

GR Assignments 04

1. Tensor Analysis III: the Covariant Divergence and the Laplacian

   An extremely useful identity for the variation (in particular, the derivative) of the determinant $g := |\det g_{\mu\nu}|$ of the metric is (see section 4.6 of the lecture notes)

   $$g^{-1}\delta g = g^{\mu\nu}\delta g_{\mu\nu} \qquad g^{-1}\partial_\lambda g = g^{\mu\nu}\partial_\lambda g_{\mu\nu} \quad . \tag{1}$$

   (a) Use (1) to show that the contracted Christoffel symbol $\Gamma^\mu_{\ \mu\lambda}$ (summation over the index μ) can be calculated by the simple formula

   $$\Gamma^\mu_{\ \mu\lambda} = g^{-1/2}\partial_\lambda g^{+1/2} \quad . \tag{2}$$

   (b) Show that this implies that the covariant divergence of a vector (current) J^μ and an anti-symmetric tensor $F^{\mu\nu} = -F^{\nu\mu}$ can be written as

   $$\nabla_\mu J^\mu = g^{-1/2}\partial_\mu(g^{1/2}J^\mu) \qquad \nabla_\mu F^{\mu\nu} = g^{-1/2}\partial_\mu(g^{1/2}F^{\mu\nu}) \quad . \tag{3}$$

   (c) Use the formula

   $$\Box\Phi = g^{\alpha\beta}\nabla_\alpha\nabla_\beta\Phi = \nabla_\alpha(g^{\alpha\beta}\nabla_\beta\Phi) = g^{-1/2}\partial_\alpha(g^{1/2}g^{\alpha\beta}\partial_\beta\Phi) \tag{4}$$

   (the final equality is a consequence of eq. (3)) to determine the Laplacian in $\mathbb{R}^3$ in spherical coordinates (r, θ, ϕ).

2. Freely Falling Schwarzschild Observers

   Consider a freely (and radially) falling observer in the Schwarzschild geometry, initially at rest at radius $r(\tau = 0) \equiv R > 2m$. Show that the proper time it would (formally) take him to reach $r = 0$ is (up to factors of c) given by (see section 25.2)

   $$\tau = \pi \left(\frac{R^3}{8m}\right)^{1/2} \quad . \tag{5}$$

   Estimate this for R the radius of the sun ($R \sim 7\times10^{10}$ cm) and $2m$ its Schwarzschild radius ($2m \sim 3 \times 10^5$ cm), restoring the correct factors of c, and show that this is of the order of an hour.

   Remark: this can be interpreted as a very rough estimate for the time of complete collapse of a star under its own gravitational attraction.