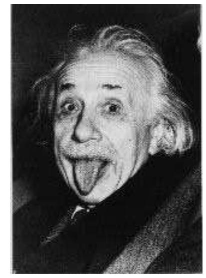

GR Assignments 04

1. Tensor Analysis II: the Covariant Divergence and the Laplacian

   An extremely useful identity for the variation (in particular, the derivative) of the determinant $g := |\det g_{\mu\nu}|$ of the metric is (see section 5.6 of the lecture notes)

   $$g^{-1}\delta g = g^{\mu\nu}\delta g_{\mu\nu} \qquad\qquad g^{-1}\partial_\lambda g = g^{\mu\nu}\partial_\lambda g_{\mu\nu} \quad . \tag{1}$$

   (a) Show that this implies that the covariant divergence of a vector (current) J^μ can be calculated by the simple formula

   $$\nabla_\mu J^\mu = g^{-1/2}\partial_\mu(g^{1/2}J^\mu) \quad , \tag{2}$$

   (b) Show that this in turn implies that the covariant divergence of an anti-symmetric tensor $F^{\mu\nu} = -F^{\nu\mu}$ and the covariant Laplacian of a scalar f, defined by $\Box f = g^{\alpha\beta}\nabla_\alpha\nabla_\beta f$, can be written as

   $$\nabla_\mu F^{\mu\nu} = g^{-1/2}\partial_\mu(g^{1/2}F^{\mu\nu}) \quad , \quad \Box f = \nabla_\alpha(g^{\alpha\beta}\nabla_\beta f) = g^{-1/2}\partial_\alpha(g^{1/2}g^{\alpha\beta}\partial_\beta f) \tag{3}$$

2. Static Schwarzschild Observers in Eddington-Finkelstein Coordinates

   In Exercise 03.2(a), you determined the acceleration of a static observer in Schwarzschild (SS) coordinates. The acceleration vector a^α appeared to be non-singular in these coordinates. This is misleading, however: its norm (a scalar) turned out to be singular as $r \to 2m$ because the relevant component $g_{rr} = f(r)^{-1}$ is singular there.

   The aim of this exercise is to look at what happens in the simplest coordinate system in which the metric is not singular at $r = 2m$, namely in Eddington-Finkelstein (EF) coordinates,

   $$ds^2 = -f(r)dv^2 + 2dvdr + r^2 d\Omega^2 \quad . \tag{4}$$

   (a) Calculate $a^\alpha = \dot{u}^\alpha + \Gamma^\alpha_{\beta\gamma}u^\beta u^\gamma$ for a static observer in EF coordinates.

   (b) Alternatively, determine a^α in EF coordinates simply from the result in SS coordinates by using the coordinate transformation.

   (c) Determine the norm $g_{\alpha\beta}a^\alpha a^\beta$ in EF coordinates, and show that the result is equal to that obtained previously in SS coordinates.