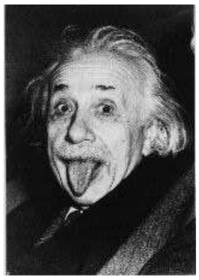

GR Assignments 05

1. On the Klein-Gordon Field in a Curved Space-Time

   The action of a real (free, massice) scalar field ϕ in a gravitational background $g_{\alpha\beta}$ is

   $$S[\phi, g_{\alpha\beta}] = \int \sqrt{g} d^4x \; L \equiv -\tfrac{1}{2} \int \sqrt{g} d^4x \left(g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + m^2 \phi^2 \right) \tag{1}$$

   The resulting equation of motion is $\left(\Box - m^2\right) \phi = 0$ (make sure that you know how to derive this!), and the generally covariant energy-momentum tensor is

   $$T_{\alpha\beta} = \partial_\alpha \phi \partial_\beta \phi + g_{\alpha\beta} L \tag{2}$$

   (a) Show that $T_{\alpha\beta}$ is (covariantly) conserved when ϕ is a solution to the Klein-Gordon equation of motion, i.e.

   $$\left(\Box - m^2\right) \phi = 0 \quad \Rightarrow \quad \nabla^\alpha T_{\alpha\beta} = 0 \ . \tag{3}$$

   (b) Show that $T_{\alpha\beta}$ is related to the variation of the action *with respect to the metric* by

   $$\delta S = -\tfrac{1}{2} \int \sqrt{g} d^4x \; T_{\alpha\beta} \delta g^{\alpha\beta} \tag{4}$$

2. On the Maxwell Equations in a Curved Space-Time

   The Maxwell action in a gravitational background is

   $$S[A_\alpha, g_{\alpha\beta}] = \int \sqrt{g} d^4x \; L = -\tfrac{1}{4} \int \sqrt{g} d^4x F_{\alpha\beta} F^{\alpha\beta} \tag{5}$$

   The vacuum Maxwell equations are $\nabla_\alpha F^{\alpha\beta} = 0, \nabla_{[\alpha} F_{\beta\gamma]} = 0$ (make sure that you know how to derive these!), and the gauge-invariant and generally covariant energy momentum tensor is

   $$T_{\alpha\beta} = F_{\alpha\gamma} F_\beta{}^\gamma - \tfrac{1}{4} g_{\alpha\beta} F_{\gamma\delta} F^{\gamma\delta} \tag{6}$$

   (a) Use the Maxwell equations to deduce the covariant conservation law

   $$\nabla_\alpha F^{\alpha\beta} = 0 \quad , \quad \nabla_\alpha F_{\beta\gamma} + \text{cyclic permutations} = 0 \quad \Rightarrow \quad \nabla^\alpha T_{\alpha\beta} = 0 \ . \tag{7}$$

   Remark: This is a tensorial equation. For such calculations you should just use the properties of the covariant derivative and **not** write out the covariant derivative in terms of the non-tensorial Christoffel symbols and partial derivatives.

Hint: Instead of embarking blindly on this calculation, remind yourself first how to do the calculation in Minkowski space. Exactly the same procedure should then work in general. If done correctly, this should be a four-line calculation.

(b) Show that the energy momentum tensor is related to the variation of the Maxwell action with respect to the metric in the same way as for the scalar field in (4).

Hint: don't forget the implicit metric-dependence in expressions like $F_{\alpha\beta}F^{\alpha\beta}$.

3. PAINLEVÉ-GULLSTRAND COORDINATES FOR THE SCHWARZSCHILD SPACE-TIME

In the Schwarzschild coordinates (t, r), the Schwarzschild metric has the standard form

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2 d\Omega^2 \qquad f(r) = 1 - \frac{2m}{r} \; . \tag{8}$$

(a) Show that the metric

$$ds^2 = -f(r)dT^2 + 2C(r)dTdr + f(r)^{-1}(1 - C(r)^2)dr^2 + r^2 d\Omega^2 \tag{9}$$

is equivalent to the Schwarzschild metric for *any* function $C(r)$. [**Hint:** Begin with (8) and consider the coordinate transformation $T(t, r) = t + \psi(r)$.]

(b) Now choose $C(r)$ such that $g_{rr} = 1$ (Painlevé-Gullstrand (PG) coordinates). Write down the resulting metric and show that it is completely non-singular for all $r > 0$ (in particular for $r \to 2m$), i.e. show that the metric coefficients are bounded and the determinant is non-zero.

(c) Show that the choice $C(r) = 1$ gives rise to the metric in Eddington-Finkelstein coordinates (with $T \equiv v = t + r^*$).

Optional Further Exercises:

Test your understanding/knowledge of GR

(solutions will *not* be provided - see the Lecture notes for details).

The metric in PG coordinates is related to timelike geodesics in the same way as the metric in Eddington-Finkelstein coordinates is related to null geodesics. To see this, consider the field of normal vectors $u_\alpha = -\partial_\alpha T$ orthogonal to the surfaces of constant T (in Scharzschild coordinates $x^\alpha = (t, r, \ldots)$).

(d) Show that $u^\alpha u_\alpha = -1$. Then show that in general the two properties $u^\alpha u_\alpha =$ const and $u_\alpha = -\partial_\alpha T$ imply that u^α is geodesic, i.e. $u^\beta \nabla_\beta u^\alpha = 0$.

(e) Show that the geodesics $x^\alpha(\tau)$ to which the u^α are tangent ($u^\alpha = \dot{x}^\alpha$) are radial geodesics ($L = 0$) with proper time $\tau = T$ and energy $E = 1$ (corresponding to observers that would have started off at rest at $r = \infty$).