GR Assignments 03

1. Tensor Algebra

   Let $f(x)$ be a scalar field, $V^\alpha(x)$ a vector field, $A_\alpha(x)$ a covector field, and denote by $\partial_\alpha = \partial/\partial x^\alpha$ the partial derivatives.

   (a) Show that $\partial_\alpha f$ is (i.e. transforms like) a covector field.

   (b) Show that the first-order linear differential operator

   $$V(x) = V^\alpha(x)\,\partial_\alpha \tag{1}$$

   is *invariant* under coordinate transformations.

   (c) Analogously, show that

   $$A(x) = A_\alpha(x) dx^\alpha \tag{2}$$

   is invariant under coordinate transformations.

   Remark: It is extremely useful to think of vector fields in this way. The basic *coordinate-independent* object is V. V can be expanded in a basis ∂_α, and its components with respect to this basis are the V^α. If you change coordinates, the basis changes, and therefore also the components of V change when expanded with respect to this new basis.

   Likewise, the argument shows that covector fields are most naturally (and invariantly) to be thought of as objects that can (and want to) be integrated over one-dimensional curves, with $\oint A(x) = \oint A_\alpha dx^\alpha$ independent of the choice of coordinates.

2. The Effective Geodesic Potential

   Generalising the discussion in section 25.3 of the lecture notes, and following Remark 6 at the end of that section, derive the effective potential equation for the general class of static spherically symmetric metrics of the form

   $$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2 d\Omega^2 \quad , \quad f(r) = 1 + 2\phi(r) \tag{3}$$

   (this includes e.g. possibly electrically and / or magnetically charged stars and black holes, and / or in the presence of a cosmological constant).