GR Assignments 04

1. Tensor Analysis I: the Covariant Derivative

   The covariant derivatives of a covector field A_μ and a $(0,2)$ tensor $B_{\mu\nu}$ field are

$$\begin{aligned} \nabla_\mu A_\nu &= \partial_\mu A_\nu - \Gamma^\lambda_{\mu\nu} A_\lambda \\ \nabla_\lambda B_{\mu\nu} &= \partial_\lambda B_{\mu\nu} - \Gamma^\rho{}_{\lambda\mu} B_{\rho\nu} - \Gamma^\rho{}_{\lambda\nu} B_{\mu\rho} \ . \end{aligned} \tag{1}$$

   (a) Show that, even though $\partial_\mu A_\nu$ is *not* a tensor (which is why the Γ-term is required in $\nabla_\mu A_\nu$), the *curl* (or *rotation*) $\partial_\mu A_\nu - \partial_\nu A_\mu$ *is* (i.e. transforms as) a tensor. Then show that the covariant curl of a covector is equal to its ordinary curl,

$$\nabla_\mu A_\nu - \nabla_\nu A_\mu = \partial_\mu A_\nu - \partial_\nu A_\mu \ . \tag{2}$$

   This provides an alternative argument for the fact that $\partial_\mu A_\nu - \partial_\nu A_\mu$ is a tensor.

   (b) Show that the covariant derivative of the metric is zero, $\nabla_\lambda g_{\mu\nu} = 0$.

2. Stationary and Freely Falling Schwarzschild Observers

   (a) Consider a stationary observer (sitting at fixed values of $(r > 2m, \theta, \phi)$) in the Schwarzschild geometry

$$ds^2 = -\left(1 - \frac{2m}{r}\right) dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \ . \tag{3}$$

   Determine his worldline 4-velocity $u^\alpha = dx^\alpha/d\tau$ and the covariant (tensorial, specifically vectorial) acceleration $a^\alpha = \dot{u}^\alpha + \Gamma^\alpha_{\beta\gamma} u^\beta u^\gamma$ and calculate $g_{\alpha\beta} a^\alpha a^\beta$.

   (b) Consider a freely (and radially) falling observer in the Schwarzschild geometry, initially at rest at radius $r(\tau = 0) \equiv R > 2m$. Show that the proper time it would (formally) take him to reach $r = 0$ is (up to factors of c) given by

$$\tau = \pi \left(\frac{R^3}{8m}\right)^{1/2} \ . \tag{4}$$

   Estimate this for R the radius of the sun ($R \sim 7 \times 10^{10}$ cm) and $2m$ its Schwarzschild radius ($2m \sim 3 \times 10^5$ cm), restoring the correct factors of c, and show that this is of the order of an hour.

   Remark: this can be interpreted as an estimate for the time of complete collapse of a star under its own gravitational attraction.