GR Assignments 05

1. Tensor Analysis II: the Covariant Divergence and the Laplacian

   An extremely useful identity for the variation (in particular, the derivative) of the determinant $g := |\det g_{\mu\nu}|$ of the metric is (see section 5.6 of the lecture notes)

   $$g^{-1}\delta g = g^{\mu\nu}\delta g_{\mu\nu} \qquad\qquad g^{-1}\partial_\lambda g = g^{\mu\nu}\partial_\lambda g_{\mu\nu} \quad . \tag{1}$$

   (a) Show that this implies that the covariant divergence of a vector (current) J^μ can be calculated by the simple formula

   $$\nabla_\mu J^\mu = g^{-1/2}\partial_\mu(g^{1/2}J^\mu) \quad , \tag{2}$$

   (b) Show that this in turn implies that the covariant divergence of an anti-symmetric tensor $F^{\mu\nu} = -F^{\nu\mu}$ and the covariant Laplacian of a scalar f, defined by $\Box f = g^{\alpha\beta}\nabla_\alpha\nabla_\beta f$, can be written as

   $$\nabla_\mu F^{\mu\nu} = g^{-1/2}\partial_\mu(g^{1/2}F^{\mu\nu}) \quad , \quad \Box f = \nabla_\alpha(g^{\alpha\beta}\nabla_\beta f) = g^{-1/2}\partial_\alpha(g^{1/2}g^{\alpha\beta}\partial_\beta f) \tag{3}$$

   (c) Use the above formula for $\Box f$ to determine the Laplace operator Δ on the unit 2-sphere, with its standard line element $ds^2 = d\Omega^2$, and in $\mathbb{R}^3$ in spherical coordinates (r, θ, ϕ).

2. Static Schwarzschild Observers in Eddington-Finkelstein Coordinates

   In Exercise 04.2(a), you determined the acceleration of a static observer in Schwarzschild (SS) coordinates. The acceleration vector a^α appeared to be non-singular in these coordinates. This is misleading, however: its norm (a scalar) turned out to be singular as $r \to 2m$ because the relevant component $g_{rr} = f(r)^{-1}$ of the Schwarzschild metric in Schwarschild coordinates is singular there.

   The aim of this exercise is to look at what happens in the simplest coordinate system in which the metric is not singular at $r = 2m$, namely in Eddington-Finkelstein (EF) coordinates,

   $$ds^2 = -f(r)dv^2 + 2dvdr + r^2 d\Omega^2 \quad . \tag{4}$$

   (a) Calculate $a^\alpha = \dot{u}^\alpha + \Gamma^\alpha_{\beta\gamma}u^\beta u^\gamma$ for a static observer in EF coordinates.

   (b) Alternatively, determine a^α in EF coordinates simply from the result in SS coordinates by using the coordinate transformation.

   (c) Determine the norm $g_{\alpha\beta}a^\alpha a^\beta$ in EF coordinates, and show that the result is equal to that obtained previously in SS coordinates.