GR Assignments 07

1. Riemann Curvature Tensor and Differential Identities

   In the course, we defined the Riemann curvature tensor via the commutator of covariant derivatives,

   $$[\nabla_\alpha, \nabla_\beta]V^\gamma = R^\gamma_{\ \delta\alpha\beta}V^\delta \quad , \quad [\nabla_\alpha, \nabla_\beta]T^{\gamma\delta} = R^\gamma_{\ \epsilon\alpha\beta}T^{\epsilon\delta} + R^\delta_{\ \epsilon\alpha\beta}T^{\gamma\epsilon} \quad , \tag{1}$$

   etc. and we defined its contractions, the Ricci tensor $R_{\alpha\beta} = R^\gamma_{\ \alpha\gamma\beta}$ and the Ricci scalar $R = g^{\alpha\beta}R_{\alpha\beta}$. The Riemann curvature tensor has the symmetries

   $$R_{\alpha\beta\gamma\delta} = -R_{\alpha\beta\delta\gamma} = -R_{\beta\alpha\gamma\delta} \quad , \quad R_{\alpha[\beta\gamma\delta]} = 0 \ \Leftrightarrow \ R_{\alpha\beta\gamma\delta} + R_{\alpha\gamma\delta\beta} + R_{\alpha\delta\beta\gamma} = 0 \tag{2}$$

   These symmetries also imply that the Ricci tensor is symmetric, $R_{\alpha\beta} = R_{\beta\alpha}$. [see section 8.3 of the lecture notes for proofs and make sure that you understand the details!]

   (a) Like any linear operator, the covariant derivative ∇_α satisfies the Jacobi identity

   $$[\nabla_\alpha, [\nabla_\beta, \nabla_\gamma]] + \text{cyclic permutations} = 0 \ . \tag{3}$$

   Show that this implies the *Bianchi identity*

   $$\nabla_\alpha R_{\mu\nu\beta\gamma} + \text{cyclic permutations in } (\alpha, \beta, \gamma) = 0 \tag{4}$$

   (b) By double-contraction of the Bianchi identity, deduce the *contracted Bianchi identity*

   $$\nabla_\alpha(2R^\alpha_{\ \gamma} - \delta^\alpha_{\ \gamma}R) = 0 \tag{5}$$

   and show that this is equivalent to the statement that the *Einstein tensor* $G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R$ has vanishing covariant divergence, $\nabla^\alpha G_{\alpha\beta} = 0$.

   (c) By double-contracting the expression for $[\nabla_\alpha, \nabla_\beta]T^{\gamma\delta}$, show that for any tensor $T^{\alpha\beta}$ one has

   $$[\nabla_\alpha, \nabla_\beta]T^{\alpha\beta} = 0 \ . \tag{6}$$

   (d) Use this result to show that the Maxwell equations $\nabla_\alpha F^{\alpha\beta} = -J^\beta$ imply that the current is covariantly conserved,

   $$\nabla_\alpha F^{\alpha\beta} = -J^\beta \quad \Rightarrow \quad \nabla_\beta J^\beta = 0 \ . \tag{7}$$

   Remark: In all these equations indices are lowered and raised with the metric and its inverse: $R_{\alpha\beta\gamma\delta} = g_{\alpha\lambda}R^\lambda_{\ \beta\gamma\delta}$, $\nabla^\alpha = g^{\alpha\rho}\nabla_\rho$, $R^\alpha_{\ \gamma} = g^{\alpha\beta}R_{\beta\gamma}$ etc.

2. CURVATURE OF A CLASS OF 2-DIMENSIONAL METRICS

Consider the 2-dimensional line element

$$ds^2 = dx^2 + f(x)^2 d\phi^2 \ . \tag{8}$$

Show that the Ricci tensor and Ricci scalar curvature of this metric are

$$R_{\alpha\beta} = -(f''/f) g_{\alpha\beta} \quad , \quad R(x) = -2f''(x)/f(x) \ . \tag{9}$$

Hint: Calculate the Christoffel symbols from the Euler-Lagrange equations. Then determine the one independent component of the Riemann tensor, e.g. $R^x{}_{\phi x \phi}$, and then deduce the Ricci tensor and Ricci scalar from this.

Remark: In particular, for the Euclidean metric and the standard metrics on the sphere and the hyperboloid one finds

$$f(x) = \begin{cases} x & (R^2) \\ \sin x & (S^2) \\ \sinh x & (H^2) \end{cases} \quad \Rightarrow \quad R = \begin{cases} 0 \\ +2 \\ -2 \end{cases} \tag{10}$$

In 2 dimensions, R is related to the *Gauss Curvature* K of a surface by $K = R/2$ so that $K = 0, \pm 1$ in these examples.