

SOLUTIONS TO ASSIGNMENTS 01

1. COORDINATE TRANSFORMATIONS AND METRICS

- (a) If we express $d\xi^A$ in terms of the new coordinates we find that $d\xi^A = \frac{\partial \xi^A}{\partial x^\mu} dx^\mu$, plugging this into the invariant line element yield the desired result :

$$ds^2 = \eta_{AB} d\xi^A d\xi^B = \eta_{AB} \frac{\partial \xi^A}{\partial x^\mu} \frac{\partial \xi^B}{\partial x^\nu} dx^\mu dx^\nu = g_{\mu\nu} dx^\mu dx^\nu \quad (1)$$

- (b) There are (at least) 2 ways to do this calculation. The longer one is to use the result of (a) to compute the components of the metric in the new coordinates one by one,

$$\begin{aligned} g_{TT} &= \eta_{AB} \frac{\partial \xi^A}{\partial T} \frac{\partial \xi^B}{\partial T} = - \left(\frac{\partial t}{\partial T} \right)^2 + \left(\frac{\partial x}{\partial T} \right)^2 \\ &= -X^2 \cosh(T)^2 + X^2 \sinh(T)^2 = -X^2 \end{aligned} \quad (2)$$

$$\begin{aligned} g_{TX} &= \eta_{AB} \frac{\partial \xi^A}{\partial T} \frac{\partial \xi^B}{\partial X} = -\frac{\partial t}{\partial T} \frac{\partial t}{\partial X} + \frac{\partial x}{\partial T} \frac{\partial x}{\partial X} \\ &= -X \cosh(T) \sinh(T) + X \sinh(T) \cosh(T) = 0 \end{aligned} \quad (3)$$

$$\begin{aligned} g_{XX} &= \eta_{AB} \frac{\partial \xi^A}{\partial X} \frac{\partial \xi^B}{\partial X} = - \left(\frac{\partial t}{\partial X} \right)^2 + \left(\frac{\partial x}{\partial X} \right)^2 \\ &= -\sinh(T)^2 + \cosh(T)^2 = 1 \end{aligned} \quad (4)$$

to conclude that the metric in Rindler coordinates is

$$ds^2 = -X^2 dT^2 + dX^2 \quad (5)$$

Alternatively (and this is frequently the calculationally more efficient way of proceeding, even though in the present example it makes hardly any difference), one can simply calculate dt and dx in terms of the new coordinates once and for all, and then plug the result into the line element to read off the components of the metric:

$$\begin{aligned} t &= X \sinh T \quad , \quad x = X \cosh T \\ \Rightarrow dt &= dX \sinh T + X \cosh T dT \quad , \quad dx = dX \cosh T + X \sinh T dT \\ \Rightarrow -dt^2 + dx^2 &= -(dX \sinh T + X \cosh T dT)^2 + (dX \cosh T + X \sinh T dT)^2 \\ &= -X^2 dT^2 + dX^2 \quad . \end{aligned} \quad (6)$$

- (c) We show that $g^{\mu\nu}$ is the inverse of $g_{\mu\nu}$ by checking that $g_{\mu\nu}g^{\nu\rho} = \delta_\mu^\rho$. To show this we will use the fact that $\frac{\partial\xi^B}{\partial x^\nu}\frac{\partial x^\nu}{\partial\xi^C} = \delta_C^B$.

$$\begin{aligned} g_{\mu\nu}g^{\nu\rho} &= \eta_{AB}\frac{\partial\xi^A}{\partial x^\mu}\frac{\partial\xi^B}{\partial x^\nu}\eta^{CD}\frac{\partial x^\nu}{\partial\xi^C}\frac{\partial x^\rho}{\partial\xi^D} = \eta_{AB}\frac{\partial\xi^A}{\partial x^\mu}\delta_C^B\eta^{CD}\frac{\partial x^\rho}{\partial\xi^D} \\ &= \eta_{AB}\eta^{BD}\frac{\partial\xi^A}{\partial x^\mu}\frac{\partial x^\rho}{\partial\xi^D} = \delta_A^D\frac{\partial\xi^A}{\partial x^\mu}\frac{\partial x^\rho}{\partial\xi^D} = \frac{\partial\xi^A}{\partial x^\mu}\frac{\partial x^\rho}{\partial\xi^A} = \delta_\mu^\rho \end{aligned} \quad (7)$$

Note that because the metric is also used to raise or lower indices we therefore have : $g_{\mu\nu}g^{\nu\rho} = g_\mu^\rho = \delta_\mu^\rho$.

2. THE FREE PARTICLE IN ARBITRARY COORDINATES

We plug the metric and the inverse metric into $\Gamma_{\nu\lambda}^\mu$ to check that :

$$\Gamma_{\nu\lambda}^\mu = \frac{1}{2}g^{\mu\rho}(g_{\rho\nu,\lambda} + g_{\rho\lambda,\nu} - g_{\lambda\nu,\rho}) \quad (8)$$

$$= \frac{1}{2}\eta^{AB}\frac{\partial x^\mu}{\partial\xi^A}\frac{\partial x^\rho}{\partial\xi^B}\left(\frac{\partial}{\partial x^\lambda}\left(\eta^{CD}\frac{\partial\xi^C}{\partial x^\rho}\frac{\partial\xi^D}{\partial x^\nu}\right) + \lambda \leftrightarrow \nu - \lambda \leftrightarrow \rho\right) \quad (9)$$

$$= \frac{\eta^{AB}}{2}\frac{\partial x^\mu}{\partial\xi^A}\frac{\partial x^\rho}{\partial\xi^B}\left(\eta^{CD}\left[\frac{\partial\xi^D}{\partial x^\nu}\frac{\partial^2\xi^C}{\partial x^\lambda\partial x^\rho} + \frac{\partial\xi^C}{\partial x^\rho}\frac{\partial^2\xi^D}{\partial x^\lambda\partial x^\nu}\right] + \lambda \leftrightarrow \nu - \lambda \leftrightarrow \rho\right) \quad (10)$$

$$= \frac{1}{2}\eta^{AB}\frac{\partial x^\mu}{\partial\xi^A}\frac{\partial x^\rho}{\partial\xi^B}\left(2\eta^{CD}\frac{\partial\xi^C}{\partial x^\rho}\frac{\partial^2\xi^D}{\partial x^\lambda\partial x^\nu}\right) \quad (11)$$

$$= \frac{\partial x^\mu}{\partial\xi^A}\left(\underbrace{\eta^{AB}\delta_B^C\eta^{CD}}_{=\delta_D^A}\frac{\partial^2\xi^D}{\partial x^\lambda\partial x^\nu}\right) = \frac{\partial x^\mu}{\partial\xi^A}\frac{\partial^2\xi^A}{\partial x^\lambda\partial x^\nu} \quad (12)$$

3. CHRISTOFFEL SYMBOLS AND COORDINATE TRANSFORMATIONS

- (a) We want to show how the Christoffel symbols transform under a general change of coordinate of the form $x^\mu \rightarrow y^{\mu'}(x)$, more precisely we need to compute $\Gamma_{\nu'\lambda'}^{\mu'}$ in terms of the x^μ coordinate. As we can see below, the first step will be to understand how partial derivative of the metric transforms.

$$\frac{1}{2}g^{\mu\rho}(\partial_\lambda g_{\rho\nu} + \partial_\nu g_{\rho\lambda} - \partial_\rho g_{\nu\lambda}) \rightarrow \frac{1}{2}g'^{\mu'\rho'}(\partial_{\lambda'} g'_{\rho'\nu'} + \partial_{\nu'} g'_{\rho'\lambda'} - \partial_{\rho'} g'_{\nu'\lambda'}) = \Gamma_{\nu'\lambda'}^{\mu'} \quad (13)$$

For this purpose, we have to use the fact that the metric is a rank 2 tensor. This specifies the way in which it will transform under the change of coordinate :

$$g_{\mu\nu} \rightarrow g'_{\mu'\nu'} = \frac{\partial x^\mu}{\partial y^{\mu'}}\frac{\partial x^\nu}{\partial y^{\nu'}}g_{\mu\nu} \quad g^{\mu\nu} \rightarrow g'^{\mu'\nu'} = \frac{\partial y^{\mu'}}{\partial x^\mu}\frac{\partial y^{\nu'}}{\partial x^\nu}g^{\mu\nu} \quad (14)$$

We also recall that partial derivative are covariant vectors, which means that they transform like :

$$\partial_\lambda \rightarrow \partial_{\lambda'} = \frac{\partial x^\lambda}{\partial y^{\lambda'}}\partial_\lambda \quad (15)$$

Using (13) and (14), we are able to determine how partial derivative of the metric transforms :

$$\begin{aligned}\partial_\lambda g_{\mu\nu} \rightarrow \partial_{\lambda'} g'_{\mu'\nu'} &= \frac{\partial x^\lambda}{\partial y^{\lambda'}} \partial_\lambda \frac{\partial x^\mu}{\partial y^{\mu'}} \frac{\partial x^\nu}{\partial y^{\nu'}} g_{\mu\nu} \\ &= \frac{\partial x^\lambda}{\partial y^{\lambda'}} \frac{\partial x^\mu}{\partial y^{\mu'}} \frac{\partial x^\nu}{\partial y^{\nu'}} \partial_\lambda g_{\mu\nu} + g_{\mu\nu} \frac{\partial x^\lambda}{\partial y^{\lambda'}} \frac{\partial}{\partial x^\lambda} \left(\frac{\partial x^\mu}{\partial y^{\mu'}} \frac{\partial x^\nu}{\partial y^{\nu'}} \right)\end{aligned}\quad (16)$$

where the second term can be developed as :

$$\begin{aligned}g_{\mu\nu} \frac{\partial x^\lambda}{\partial y^{\lambda'}} \frac{\partial}{\partial x^\lambda} \left(\frac{\partial x^\mu}{\partial y^{\mu'}} \frac{\partial x^\nu}{\partial y^{\nu'}} \right) &= g_{\mu\nu} \frac{\partial}{\partial y^{\lambda'}} \left(\frac{\partial x^\mu}{\partial y^{\mu'}} \frac{\partial x^\nu}{\partial y^{\nu'}} \right) \\ &= g_{\mu\nu} \left(\frac{\partial x^\nu}{\partial y^{\nu'}} \frac{\partial^2 x^\mu}{\partial y^{\lambda'} \partial y^{\mu'}} + \frac{\partial x^\mu}{\partial y^{\mu'}} \frac{\partial^2 x^\nu}{\partial y^{\lambda'} \partial y^{\nu'}} \right)\end{aligned}\quad (17)$$

If we use the fact that the metric is symmetric : $g_{\mu\nu} = g_{\nu\mu}$, we get :

$$\partial_{\lambda'} g'_{\mu'\nu'} = \frac{\partial x^\lambda}{\partial y^{\lambda'}} \frac{\partial x^\mu}{\partial y^{\mu'}} \frac{\partial x^\nu}{\partial y^{\nu'}} \partial_\lambda g_{\mu\nu} + g_{\mu\nu} \left(\frac{\partial x^\nu}{\partial y^{\nu'}} \frac{\partial^2 x^\mu}{\partial y^{\lambda'} \partial y^{\mu'}} + \frac{\partial x^\mu}{\partial y^{\mu'}} \frac{\partial^2 x^\nu}{\partial y^{\lambda'} \partial y^{\nu'}} \right) \quad (18)$$

Using this expression three times and relabeling the indices, one can write :

$$\begin{aligned}(\partial_{\lambda'} g'_{\rho'\nu'} + \partial_{\nu'} g'_{\rho'\lambda'} - \partial_{\rho'} g'_{\nu'\lambda'}) &= \frac{\partial x^\lambda}{\partial y^{\lambda'}} \frac{\partial x^\rho}{\partial y^{\rho'}} \frac{\partial x^\nu}{\partial y^{\nu'}} \left(\partial_\lambda g_{\rho\nu} + \partial_\nu g_{\rho\lambda} - \partial_\rho g_{\nu\lambda} \right) \\ &\quad + g_{\rho\nu} \left(\frac{\partial x^\nu}{\partial y^{\nu'}} \frac{\partial^2 x^\rho}{\partial y^{\lambda'} \partial y^{\rho'}} + \frac{\partial x^\nu}{\partial y^{\lambda'}} \frac{\partial^2 x^\rho}{\partial y^{\nu'} \partial y^{\rho'}} \right. \\ &\quad \left. + 2 \frac{\partial x^\nu}{\partial y^{\rho'}} \frac{\partial^2 x^\rho}{\partial y^{\lambda'} \partial y^{\nu'}} - \frac{\partial x^\rho}{\partial y^{\lambda'}} \frac{\partial^2 x^\nu}{\partial y^{\rho'} \partial y^{\nu'}} - \frac{\partial x^\rho}{\partial y^{\nu'}} \frac{\partial^2 x^\nu}{\partial y^{\rho'} \partial y^{\lambda'}} \right)\end{aligned}\quad (19)$$

But again, because the metric is symmetric in ρ and ν we are just left with :

$$\begin{aligned}(\partial_{\lambda'} g'_{\rho'\nu'} + \partial_{\nu'} g'_{\rho'\lambda'} - \partial_{\rho'} g'_{\nu'\lambda'}) &= \frac{\partial x^\lambda}{\partial y^{\lambda'}} \frac{\partial x^\rho}{\partial y^{\rho'}} \frac{\partial x^\nu}{\partial y^{\nu'}} \left(\partial_\lambda g_{\rho\nu} + \partial_\nu g_{\rho\lambda} - \partial_\rho g_{\nu\lambda} \right) \\ &\quad + 2 g_{\rho\nu} \frac{\partial x^\nu}{\partial y^{\nu'}} \frac{\partial^2 x^\rho}{\partial y^{\lambda'} \partial y^{\rho'}}\end{aligned}\quad (20)$$

Now substituting this last expression together with (13) in (12), we finally get the desired result :

$$\begin{aligned}\Gamma_{\nu'\lambda'}^{\mu'} &= \frac{1}{2} \left(\frac{\partial y^{\mu'}}{\partial x^\mu} \frac{\partial y^{\rho'}}{\partial x^\rho} g^{\mu\rho} \right) \cdot \\ &\quad \left(\frac{\partial x^\lambda}{\partial y^{\lambda'}} \frac{\partial x^\rho}{\partial y^{\rho'}} \frac{\partial x^\nu}{\partial y^{\nu'}} \left(\partial_\lambda g_{\rho\nu} + \partial_\nu g_{\rho\lambda} - \partial_\rho g_{\nu\lambda} \right) + 2 g_{\rho\nu} \frac{\partial x^\nu}{\partial y^{\nu'}} \frac{\partial^2 x^\rho}{\partial y^{\lambda'} \partial y^{\rho'}} \right) \\ &= \frac{\partial y^{\mu'}}{\partial x^\mu} \frac{\partial x^\lambda}{\partial y^{\lambda'}} \frac{\partial x^\nu}{\partial y^{\nu'}} \frac{1}{2} g^{\mu\rho} \left(\partial_\lambda g_{\rho\nu} + \partial_\nu g_{\rho\lambda} - \partial_\rho g_{\nu\lambda} \right) + \frac{\partial y^{\mu'}}{\partial x^\mu} \delta_\rho^\nu \delta_\nu^\mu \frac{\partial^2 x^\rho}{\partial y^{\lambda'} \partial y^{\nu'}} \\ &= \frac{\partial y^{\mu'}}{\partial x^\mu} \frac{\partial x^\lambda}{\partial y^{\lambda'}} \frac{\partial x^\nu}{\partial y^{\nu'}} \Gamma_{\nu\lambda}^\mu + \frac{\partial y^{\mu'}}{\partial x^\mu} \frac{\partial^2 x^\mu}{\partial y^{\lambda'} \partial y^{\nu'}}\end{aligned}\quad (21)$$

(b) As we can see, \dot{x}^μ is a vector and therefore transforms like :

$$\dot{x}^\mu = \frac{d}{d\tau} x^\mu \rightarrow \frac{d}{d\tau} y^{\mu'} = \frac{\partial y^{\mu'}}{\partial x^\mu} \frac{d}{d\tau} x^\mu = \frac{\partial y^{\mu'}}{\partial x^\mu} \dot{x}^\mu \quad (22)$$

But \ddot{x}^μ is not :

$$\begin{aligned} \ddot{x}^\mu &= \frac{d}{d\tau} \dot{x}^\mu \rightarrow \frac{d}{d\tau} \dot{y}^{\mu'} &= \frac{d}{d\tau} \left(\frac{\partial y^{\mu'}}{\partial x^\mu} \dot{x}^\mu \right) &= \frac{\partial y^{\mu'}}{\partial x^\mu} \ddot{x}^\mu + \frac{\partial^2 y^{\mu'}}{\partial x^\mu \partial x^\nu} \dot{x}^\mu \dot{x}^\nu \\ &= \frac{\partial y^{\mu'}}{\partial x^\mu} \left(\ddot{x}^\mu + \frac{\partial x^\mu}{\partial y^{\rho'}} \frac{\partial^2 y^{\rho'}}{\partial x^\lambda \partial x^\nu} \dot{x}^\lambda \dot{x}^\nu \right) \end{aligned} \quad (23)$$

Still, if we look at the transformation of the following combination $\ddot{x}^\mu + \Gamma_{\nu\lambda}^\mu \dot{x}^\nu \dot{x}^\lambda \rightarrow \ddot{y}^{\mu'} + \Gamma_{\nu'\lambda'}^{\mu'} \dot{y}^{\nu'} \dot{y}^{\lambda'}$, it turns out to be a vector as we will see. Using (20), (21) and (22) one can write :

$$\begin{aligned} \ddot{y}^{\mu'} + \Gamma_{\nu'\lambda'}^{\mu'} \dot{y}^{\nu'} \dot{y}^{\lambda'} &= \frac{\partial y^{\mu'}}{\partial x^\mu} \ddot{x}^\mu + \frac{\partial^2 y^{\mu'}}{\partial x^\mu \partial x^\nu} \dot{x}^\mu \dot{x}^\nu \\ &+ \left[\frac{\partial y^{\mu'}}{\partial x^\mu} \frac{\partial x^\lambda}{\partial y^{\lambda'}} \frac{\partial x^\nu}{\partial y^{\nu'}} \Gamma_{\nu\lambda}^\mu + \frac{\partial y^{\mu'}}{\partial x^\mu} \frac{\partial^2 x^\mu}{\partial y^{\lambda'} \partial y^{\nu'}} \right] \frac{\partial y^{\nu'}}{\partial x^\nu} \dot{x}^\nu \frac{\partial y^{\lambda'}}{\partial x^\lambda} \dot{x}^\lambda \end{aligned} \quad (24)$$

which is :

$$= \frac{\partial y^{\mu'}}{\partial x^\mu} \left[\ddot{x}^\mu + \Gamma_{\nu\lambda}^\mu \dot{x}^\nu \dot{x}^\lambda + \frac{\partial x^\mu}{\partial y^{\rho'}} \frac{\partial^2 y^{\rho'}}{\partial x^\lambda \partial x^\nu} \dot{x}^\lambda \dot{x}^\nu + \frac{\partial^2 x^\mu}{\partial y^{\lambda'} \partial y^{\nu'}} \frac{\partial y^{\nu'}}{\partial x^\nu} \frac{\partial y^{\lambda'}}{\partial x^\lambda} \dot{x}^\nu \dot{x}^\lambda \right] \quad (25)$$

Now to show that the two last terms indeed cancel out, we compute the partial derivative of $\frac{\partial y^{\rho'}}{\partial x^\nu} \frac{\partial x^\mu}{\partial y^{\rho'}} = \delta_\nu^\mu$, one obtains :

$$\begin{aligned} 0 &= \frac{\partial}{\partial x^\lambda} \left(\frac{\partial y^{\rho'}}{\partial x^\nu} \frac{\partial x^\mu}{\partial y^{\rho'}} \right) &= \frac{\partial x^\mu}{\partial y^{\rho'}} \frac{\partial^2 y^{\rho'}}{\partial x^\lambda \partial x^\nu} + \frac{\partial y^{\rho'}}{\partial x^\nu} \frac{\partial}{\partial x^\lambda} \frac{\partial x^\mu}{\partial y^{\rho'}} \\ &= \frac{\partial x^\mu}{\partial y^{\rho'}} \frac{\partial^2 y^{\rho'}}{\partial x^\lambda \partial x^\nu} + \frac{\partial y^{\rho'}}{\partial x^\nu} \frac{\partial y^{\lambda'}}{\partial x^\lambda} \frac{\partial^2 x^\mu}{\partial y^{\lambda'} \partial y^{\rho'}} \end{aligned} \quad (26)$$

so that (24) gives the desired result : the combination $\ddot{x}^\mu + \Gamma_{\nu\lambda}^\mu \dot{x}^\nu \dot{x}^\lambda$ transforms like a vector.