Solutions to Assignments 02

1. Geodesics

   (a) With the Lagrangian

$$\mathcal{L}(x^\mu, \dot{x}^\mu) = \frac{1}{2} g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = \frac{1}{2} g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu \tag{1}$$

   where $g_{\mu\nu} = g_{\mu\nu}(x^\rho)$ one computes

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial x^\mu} &= \frac{1}{2} \dot{x}^\rho \dot{x}^\nu \partial_\mu g_{\rho\nu} && (2)\\
\frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} &= g_{\rho\nu} \dot{x}^\rho \frac{\partial}{\partial \dot{x}^\mu} \dot{x}^\nu = g_{\rho\nu} \dot{x}^\rho \delta^\nu_\mu = g_{\rho\mu} \dot{x}^\rho && (3)\\
\frac{d}{d\tau}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}^\mu}\right) &= g_{\rho\mu} \ddot{x}^\rho + \dot{x}^\rho \dot{x}^\nu \partial_\nu g_{\rho\mu} = g_{\rho\mu} \ddot{x}^\rho + \frac{1}{2}\left(\dot{x}^\rho \dot{x}^\nu \partial_\nu g_{\rho\mu} + \dot{x}^\nu \dot{x}^\rho \partial_\rho g_{\nu\mu}\right) && (4)
\end{aligned}$$

   Thus the Euler-Lagrange equations become

$$\begin{aligned}
\Big[\text{E.-L.}\Big] &= g_{\mu\rho} \ddot{x}^\rho + \frac{1}{2}\left(\partial_\nu g_{\rho\mu} + \partial_\rho g_{\nu\mu} - \partial_\mu g_{\rho\nu}\right) \dot{x}^\nu \dot{x}^\rho && (5)\\
&= g_{\mu\rho} \ddot{x}^\rho + \Gamma_{\mu\nu\rho} \dot{x}^\rho \dot{x}^\nu = 0 && (6)
\end{aligned}$$

   and they can be written in the usual (geodesic equation) form by multiplying by $g^{\lambda\mu}$ to move the index μ up :

$$g^{\lambda\mu}\left(g_{\mu\rho} \ddot{x}^\rho + \Gamma_{\mu\nu\rho} \dot{x}^\rho \dot{x}^\nu\right) = \ddot{x}^\lambda + \Gamma^\lambda_{\nu\rho} \dot{x}^\rho \dot{x}^\nu = 0 \tag{7}$$

   (b) First we compute :

$$\frac{d}{d\tau}\mathcal{L} = \frac{1}{2}\frac{d}{d\tau} g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu = \frac{1}{2}\Big(2 g_{\mu\nu} \ddot{x}^\mu \dot{x}^\nu + (\dot{x}^\rho \partial_\rho g_{\mu\nu}) \dot{x}^\mu \dot{x}^\nu\Big) \tag{8}$$

   Now using the identity

$$\partial_\rho g_{\mu\nu} = \Gamma_{\mu\nu\rho} + \Gamma_{\nu\mu\rho} \tag{9}$$

   together with the fact that $x^\mu(\tau)$ is a solution to the geodesic equation, which means that we also have :

$$\ddot{x^\mu} = -\Gamma^\mu_{\nu\rho} \dot{x}^\nu \dot{x}^\rho \tag{10}$$

   leaves us with

$$\begin{aligned}
\frac{d}{d\tau}\mathcal{L} &= \frac{1}{2}\Big(-2 g_{\mu\nu} \Gamma^\mu_{\lambda\rho} \dot{x}^\lambda \dot{x}^\rho \dot{x}^\nu + \dot{x}^\rho (\Gamma_{\mu\nu\rho} + \Gamma_{\nu\mu\rho}) \dot{x}^\mu \dot{x}^\nu\Big) && (11)\\
&= \frac{1}{2}\Big(-2 \Gamma_{\nu\lambda\rho} \dot{x}^\lambda \dot{x}^\rho \dot{x}^\nu + (\Gamma_{\mu\nu\rho} + \Gamma_{\nu\mu\rho}) \dot{x}^\mu \dot{x}^\nu \dot{x}^\rho\Big) = 0 && (12)
\end{aligned}$$

   which is obviously zero if we relabel the indices.

(c) The metric on the 2-sphere is

$$ds^2 = R^2\left(d\theta^2 + \sin^2(\theta)d\phi^2\right) \tag{13}$$

so that in the (θ, ϕ) coordinates we have :

$$g_{\mu\nu} = R^2\begin{pmatrix} 1 & 0 \\ 0 & \sin(\theta)^2 \end{pmatrix} \qquad g^{\mu\nu} = R^{-2}\begin{pmatrix} 1 & 0 \\ 0 & \sin(\theta)^{-2} \end{pmatrix} \tag{14}$$

Because $g_{\mu\nu}$ is diagonal and $g_{\mu\nu} = g_{\mu\nu}(\theta)$, the only non-vanishing contributions to the Christoffel symbols will come from terms involving $\partial_\theta g_{\phi\phi}$. Keeping this in mind, we first compute the Christoffel symbols with upper index θ,

$$\Gamma^\theta_{\nu\lambda} = \frac{1}{2}g^{\theta\theta}\left(\partial_\lambda g_{\theta\nu} + \partial_\nu g_{\theta\lambda} - \partial_\theta g_{\nu\lambda}\right) = -\frac{1}{2}g^{\theta\theta}\partial_\theta g_{\nu\lambda} \tag{15}$$

and we see that the only non-vanishing term with θ on the top is $\Gamma^\theta_{\phi\phi}$:

$$\Gamma^\theta_{\phi\phi} = -\frac{1}{2}g^{\theta\theta}\partial_\theta g_{\phi\phi} = -\sin(\theta)\cos(\theta) \tag{16}$$

Now if we choose ϕ to be on the top, we get

$$\Gamma^\phi_{\nu\lambda} = \frac{1}{2}g^{\phi\phi}\left(\partial_\lambda g_{\phi\nu} + \partial_\nu g_{\phi\lambda} - \partial_\phi g_{\nu\lambda}\right) = \frac{1}{2}g^{\phi\phi}\left(\partial_\lambda g_{\phi\nu} + \partial_\nu g_{\phi\lambda}\right) \tag{17}$$

so that the only non-vanishing terms with ϕ on the top are $\Gamma^\phi_{\phi\theta} = \Gamma^\phi_{\theta\phi}$:

$$\Gamma^\phi_{\phi\theta} = \frac{1}{2}g^{\phi\phi}\partial_\theta g_{\phi\phi} = \frac{\cos(\theta)}{\sin(\theta)} \tag{18}$$

With these Christoffel symbols we can now write down the geodesic equation $\ddot{x}^\mu + \Gamma^\mu_{\nu\rho}\dot{x}^\nu\dot{x}^\rho$ in the (θ, ϕ) coordinate. We find

$$\begin{cases} \ddot{\theta} + \Gamma^\theta_{\phi\phi}\dot{\phi}\dot{\phi} = \ddot{\theta} - \sin(\theta)\cos(\theta)\dot{\phi}\dot{\phi} = 0 & \mu = \theta \\ \\ \ddot{\phi} + 2\Gamma^\phi_{\theta\phi}\dot{\theta}\dot{\phi} = \ddot{\phi} + 2\dfrac{\cos(\theta)}{\sin(\theta)}\dot{\theta}\dot{\phi} = 0 & \mu = \phi \end{cases} \tag{19}$$

Using the Euler-Lagrange equations with $\mathcal{L} = \frac{1}{2}(\dot{\theta}^2 + \sin(\theta)^2\dot{\phi}^2)$, we get :

$$\begin{cases} \dfrac{d}{d\tau}\dfrac{\partial\mathcal{L}}{\partial\dot{\theta}} - \dfrac{\partial\mathcal{L}}{\partial\theta} = \dfrac{1}{2}\left(\dfrac{d}{d\tau}2\dot{\theta} - 2\sin(\theta)\cos(\theta)\dot{\phi}^2\right) = 0 \\ \\ \dfrac{d}{d\tau}\dfrac{\partial\mathcal{L}}{\partial\dot{\phi}} - \dfrac{\partial\mathcal{L}}{\partial\phi} = \dfrac{1}{2}\dfrac{d}{d\tau}\left(2\sin(\theta)^2\dot{\phi}\right) = \sin(\theta)^2\ddot{\phi} + 2\cos(\theta)\sin(\theta)\dot{\theta}\dot{\phi} = 0 \end{cases} \tag{20}$$

which are the same equations as in (19).
We can easily see that the great circles $(\theta(\tau), \phi(\tau)) = (\tau, \phi_0)$ on S^2 are solutions to the equations just found. It is indeed the case because for that particular solution ϕ is constant along a great circle, which implies $\ddot{\phi} = \dot{\phi} = 0$ and simultaneously $\theta(\tau) = \tau$ so that $\ddot{\theta}$ vanishes. By looking at equation (19) or (20) we can see that every curve with $\ddot{\theta} = \ddot{\phi} = \dot{\phi} = 0$ trivially satisfy both equations and therefore such curves are geodesics.