Solutions to Assignments 02

1. Geodesics

(a) With the Lagrangian

$$\mathcal{L}(x^\mu, \dot{x}^\mu) = \frac{1}{2} g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = \frac{1}{2} g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu$$

(1)

where $g_{\mu\nu} = g_{\mu\nu}(x^\rho)$ one computes

$$\frac{\partial \mathcal{L}}{\partial x^\mu} = \frac{1}{2} \dot{x}^\rho \dot{x}^\nu \frac{\partial g_{\mu\nu}}{\partial x^\rho}$$

(2)

$$\frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} = g_{\mu\rho} \dot{x}^\rho \frac{\partial}{\partial \dot{x}^\mu}$$

(3)

$$\frac{d}{d\tau} \left( \frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} \right) = g_{\mu\rho} \ddot{x}^\rho + \dot{x}^\rho \dot{x}^\nu \partial_\rho g_{\mu\nu} = g_{\mu\rho} \ddot{x}^\rho + \frac{1}{2} (\dot{x}^\rho \dot{x}^\nu \partial_\rho g_{\mu\nu} + \dot{x}^\nu \dot{x}^\rho \partial_\mu g_{\nu\rho})$$

(4)

Thus the Euler-Lagrange equations become

$$\begin{bmatrix} \text{E.-L.} \end{bmatrix} = g_{\mu\rho} \ddot{x}^\rho + \frac{1}{2} (\partial_\nu g_{\rho\mu} + \partial_\mu g_{\nu\rho} - \partial_\rho g_{\nu\mu}) \dot{x}^\nu \dot{x}^\rho$$

(5)

$$= g_{\mu\rho} \ddot{x}^\rho + \Gamma_{\mu\nu\rho} \dot{x}^\nu \dot{x}^\rho = 0$$

(6)

and they can be written in the usual (geodesic equation) form by multiplying by $g^{\lambda\mu}$ to move the index $\mu$ up :

$$g^{\lambda\mu} (g_{\mu\rho} \ddot{x}^\rho + \Gamma_{\mu\nu\rho} \dot{x}^\nu \dot{x}^\rho) = \ddot{x}^\lambda + \Gamma_{\nu\rho} \dot{x}^\rho \dot{x}^\nu = 0$$

(7)

(b) First we compute :

$$\frac{d}{d\tau} \mathcal{L} = \frac{1}{2} \frac{d}{d\tau} g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu = \frac{1}{2} \left( 2 g_{\mu\nu} \ddot{x}^\mu \dot{x}^\nu + (\dot{x}^\rho \partial_\rho g_{\mu\nu}) \dot{x}^\mu \dot{x}^\nu \right)$$

(8)

Now using the identity

$$\partial_\rho g_{\mu\nu} = \Gamma_{\mu\nu\rho} + \Gamma_{\nu\rho\mu}$$

(9)

together with the fact that $x^\mu(\tau)$ is a solution to the geodesic equation, which means that we also have :

$$\dot{x}^\mu = -\Gamma_{\nu\rho}^\mu \dot{x}^\nu \dot{x}^\rho$$

(10)

leaves us with

$$\frac{d}{d\tau} \mathcal{L} = \frac{1}{2} \left( -2 g_{\mu\nu} \Gamma_{\lambda\rho}^\mu \dot{x}^\lambda \dot{x}^\rho \dot{x}^\nu + \dot{x}^\rho (\Gamma_{\mu\nu\rho} + \Gamma_{\nu\mu\rho}) \dot{x}^\mu \dot{x}^\nu \right)$$

(11)

$$= \frac{1}{2} \left( -2 \Gamma_{\nu\lambda\rho} \dot{x}^\lambda \dot{x}^\rho \dot{x}^\nu + (\Gamma_{\mu\nu\rho} + \Gamma_{\nu\mu\rho}) \dot{x}^\mu \dot{x}^\nu \dot{x}^\rho \right) = 0$$

(12)

which is obviously zero if we relabel the indices.
The metric on the 2-sphere is
\[ ds^2 = R^2 \left( d\theta^2 + \sin^2(\theta) d\phi^2 \right) \]
so that in the \((\theta, \phi)\) coordinates we have:
\[ g_{\mu\nu} = R^2 \begin{pmatrix} 1 & 0 \\ 0 & \sin(\theta)^2 \end{pmatrix} \quad g^{\mu\nu} = R^{-2} \begin{pmatrix} 1 & 0 \\ 0 & \sin(\theta)^{-2} \end{pmatrix} \]
Because \(g_{\mu\nu}\) is diagonal and \(g_{\mu\nu} = g_{\mu\nu}(\theta)\), the only non-vanishing contributions to the Christoffel symbols will come from terms involving \(\partial_{\theta} g_{\phi\phi}\). Keeping this in mind, we first compute the Christoffel symbols with upper index \(\theta\),
\[ \Gamma^\theta_{\nu\lambda} = \frac{1}{2} g^{\theta\theta} (\partial_\lambda g_{\theta\nu} + \partial_\nu g_{\theta\lambda} - \partial_\theta g_{\nu\lambda}) = -\frac{1}{2} g^{\theta\theta} \partial_\theta g_{\nu\lambda} \]
and we see that the only non-vanishing term with \(\theta\) on the top is \(\Gamma^\theta_{\phi\phi}\):
\[ \Gamma^\theta_{\phi\phi} = \frac{1}{2} g^{\phi\phi} \partial_\theta g_{\phi\phi} = -\sin(\theta) \cos(\theta) \]
Now if we choose \(\phi\) to be on the top, we get
\[ \Gamma^\phi_{\nu\lambda} = \frac{1}{2} g^{\phi\phi} (\partial_\lambda g_{\phi\nu} + \partial_\nu g_{\phi\lambda} - \partial_\phi g_{\nu\lambda}) = \frac{1}{2} g^{\phi\phi} (\partial_\lambda g_{\phi\nu} + \partial_\nu g_{\phi\lambda}) \]
so that the only non-vanishing terms with \(\phi\) on the top are \(\Gamma^\phi_{\phi\theta} = \Gamma^\phi_{\theta\phi} :\)
\[ \Gamma^\phi_{\phi\theta} = \frac{1}{2} g^{\phi\phi} \partial_\theta g_{\phi\phi} = \cos(\theta) \frac{\sin(\theta)}{\sin(\theta)} \]
With these Christoffel symbols we can now write down the geodesic equation \(\ddot{x}^\mu + \Gamma^\mu_{\nu\rho} \dot{x}^\nu \dot{x}^\rho\) in the \((\theta, \phi)\) coordinate. We find
\[ \begin{cases} \ddot{\theta} + 2 \Gamma^\theta_{\phi\phi} \dot{\theta} \dot{\phi} = \ddot{\theta} - \sin(\theta) \cos(\theta) \dot{\phi} = 0 & \mu = \theta \\ \ddot{\phi} + 2 \Gamma^\phi_{\phi\theta} \dot{\theta} \dot{\phi} = \ddot{\phi} + 2 \cos(\theta) \frac{\sin(\theta)}{\sin(\theta)} \dot{\theta} \dot{\phi} = 0 & \mu = \phi \end{cases} \]
Using the Euler-Lagrange equations with \(\mathcal{L} = \frac{1}{2} (\dot{\theta}^2 + \sin(\theta)^2 \dot{\phi}^2)\), we get:
\[ \begin{cases} \frac{d}{d\tau} \frac{\partial \mathcal{L}}{\partial \dot{\theta}} - \frac{\partial \mathcal{L}}{\partial \theta} = \frac{1}{2} \left( \frac{d}{d\tau} 2 \dot{\theta} - 2 \sin(\theta) \cos(\theta) \dot{\phi}^2 \right) = 0 \\ \frac{d}{d\tau} \frac{\partial \mathcal{L}}{\partial \dot{\phi}} - \frac{\partial \mathcal{L}}{\partial \phi} = \frac{1}{2} \frac{d}{d\tau} \left( 2 \sin(\theta)^2 \dot{\phi} \right) = \sin(\theta)^2 \frac{d}{d\tau} \dot{\phi} + 2 \cos(\theta) \sin(\theta) \dot{\theta} \dot{\phi} = 0 \end{cases} \]
which are the same equations as in (19).
We can easily see that the great circles \((\theta(\tau), \phi(\tau)) = (\tau, \phi_0)\) on \(S^2\) are solutions to the equations just found. It is indeed the case because for that particular solution \(\phi\) is constant along a great circle, which implies \(\ddot{\phi} = \dot{\phi} = 0\) and simultaneously \(\theta(\tau) = \tau\) so that \(\ddot{\theta}\) vanishes. By looking at equation (19) or (20) we can see that every curve with \(\ddot{\theta} = \ddot{\phi} = \dot{\phi} = 0\) trivially satisfy both equations and therefore such curves are geodesics.