1. Tensor analysis I: Tensor Algebra

(a) Under a coordinate transformation \( x^\mu \to x'^\alpha = y^\alpha (x) \) a scalar (function) \( f(x) \) transforms as \( f \to f' \) with \( f'(y(x)) = f(x) \), while according to the chain rule the partial derivatives transform with the Jacobian as

\[
\partial_\alpha \equiv \partial_{y^\alpha} = J_\alpha^\mu \partial_\mu .
\]

Thus

\[
\partial_\alpha f'(y) = J_\alpha^\mu \partial_\mu f(x)
\]

transforms as a covector.

(b) By definition of a tensor we have (writing \( A_{\alpha\beta} \) instead of \( A'_{\alpha\beta} \) for simplicity, and suppressing the argument)

\[
A_{\alpha\beta} = J_\alpha^\mu J_\beta^\nu A_{\mu\nu} , \quad B^\beta = J_\beta^\rho B^\rho,
\]

and therefore

\[
A_{\alpha\beta} B^\beta = J_\alpha^\mu J_\beta^\nu J_\rho^\beta A_{\mu\rho} B^\rho = J_\alpha^\mu \delta_\rho^\nu A_{\mu\rho} B^\nu = J_\alpha^\mu A_{\mu\nu} B^\nu
\]

is a covector. The same kind of argument now shows that

\[
A_{\alpha\beta} B^\alpha B^\beta = J_\rho^\alpha B^\rho J_\alpha^\nu A_{\mu\rho} B^\nu = \delta_\rho^\mu B^\rho A_{\mu\nu} B^\nu = B^\mu A_{\mu\nu} B^\nu
\]

is a scalar.

(c) The invariance of \( V(x) \) under coordinate transformations follows from the fact that partial derivatives are covectors and that they are contracted with a vector to form the field \( V(x) \),

\[
V_\alpha \partial_\alpha = J_\alpha^\mu J_\alpha^\nu V_\mu \partial_\nu = \delta_\mu^\nu V_\mu \partial_\nu = V_\mu \partial_\mu .
\]

Likewise for a covector:

\[
dy_\alpha = J_\rho^\alpha dx^\rho \quad \Rightarrow \quad A_\alpha dy_\alpha = J_\rho^{\alpha \beta} A_\mu dx^\rho = A_\mu dx^\mu .
\]

2. The Effective Geodesic Potential

Starting with the metric

\[
ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2d\Omega^2 , \quad f(r) = 1 + 2\phi(r)
\]

one implements the following steps:
• the Lagrangian \(\mathcal{L}\) is conserved,
\[
-f(r)\dot{r}^2 + f(r)^{-1}\dot{r}^2 + r^2(\dot{\theta}^2 + \sin^2 \theta \dot{\phi}^2) = \epsilon \tag{9}
\]
where \(\epsilon = -1, 0\) for massive (massless) particles.

• by spherical symmetry, angular momentum is conserved, thus the motion is planar, and one can choose the coordinates such that this motion takes place in the equatorial plane \(\theta = \pi/2, \dot{\theta} = 0\), leading to
\[
-f(r)\dot{r}^2 + f(r)^{-1}\dot{r}^2 + r^2\dot{\phi}^2 = \epsilon \tag{10}
\]

• rotational and time-translational symmetry lead to the conserved quantities
\[
E = f(r)\dot{r} \quad L = r^2\dot{\phi} \tag{11}
\]
(energy and angular momentum), and using these equations to eliminate \(\dot{r}\) and \(\dot{\phi}\) from the Lagrangian, one finds
\[
-E^2 f(r)^{-1} + f(r)^{-1}\dot{r}^2 + L^2/r^2 = \epsilon \tag{12}
\]
Multiplying by \(f(r)\) and rearranging, this gives
\[
\dot{r}^2 + f(r)L^2/r^2 - \epsilon f(r) = E^2 \tag{13}
\]

• This already has the desired form of an effective Newtonian potential equation, but it is typically more useful to separate the constant (asymptotically Minkowski) part of \(f(r)\) from the rest. Thus, with \(f(r) = 1 + 2\phi(r)\) one has
\[
\frac{1}{2}\dot{r}^2 + V_{eff}(r) = E_{eff} \tag{14}
\]
where
\[
V_{eff}(r) \equiv V(r) + L^2/2r^2 = \phi(r)(-\epsilon + L^2/r^2) + L^2/2r^2 \tag{15}
\]
and
\[
E_{eff} = (E^2 + \epsilon)/2 \tag{16}
\]