Solutions to Assignments 04

1. Stationary and Freely Falling Schwarzschild Observers

   (a) The observer is sitting at fixed radius and angles, therefore his worldline 4-velocity is of the form

$$\frac{dx^\mu}{d\tau} = u^\mu = (u^t, 0, 0, 0) \quad . \tag{1}$$

   The proper time normalisation condition implies

$$u^\mu u_\mu = -1 \quad \Rightarrow \quad u^t = \frac{1}{\sqrt{1-\frac{2m}{r}}} \tag{2}$$

   (we have chosen $u^t > 0$ because the oberver evolves forward in time, $\dot{t} > 0$). The acceleration is then

$$\begin{aligned} a^\mu = \nabla_\tau u^\mu &= u^\rho \nabla_\rho u^\mu \\ &= u^t \partial_t u^\mu + u^t \Gamma^\mu_{tt} u^t \\ &= \Gamma^\mu_{tt} \frac{1}{1-\frac{2m}{r}} \\ &= -\frac{1}{2} g^{\mu\rho} \partial_\rho g_{tt} \frac{1}{1-\frac{2m}{r}} \\ \text{for } \mu \neq r \quad &= 0 \\ \text{for } \mu = r \quad &= \frac{1}{2} g^{rr} \partial_r (1-\frac{2m}{r}) \frac{1}{1-\frac{2m}{r}} \\ &= -\frac{1}{2} \partial_r \frac{2m}{r} = \frac{m}{r^2} \end{aligned} \tag{3}$$

   and therefore the norm of the acceleration is

$$\begin{aligned} g_{\mu\nu} a^\mu a^\nu &= g_{rr} a^r a^r \\ &= \frac{1}{1-\frac{2m}{r}} \frac{m^2}{r^4} \quad . \end{aligned} \tag{4}$$

   Note that this approaches the Newtonian value $(m/r^2)^2$ for $r \to \infty$, while the required acceleration to keep the stationary observer at rest diverges as $r \to 2m$.

   (b) For zero angular momentum, and with $\dot{r}_{r=R} = 0$ the effective potential equation reduces to

$$E^2 - 1 = \dot{r}^2 - \frac{2m}{r} \quad \Rightarrow \quad \dot{r}^2 = \frac{2m}{r} - \frac{2m}{R} \quad , \tag{5}$$

which integrates to

$$\tau_{R\to r_1} = -(2m)^{-1/2}\int_R^{r_1} dr\ \left(\frac{Rr}{R-r}\right)^{1/2} \quad . \tag{6}$$

This integral can be calculated in closed form, e.g. via the change of variables

$$\frac{r}{R} = \sin^2\alpha \qquad \alpha_1 \le \alpha \le \frac{\pi}{2} \quad , \tag{7}$$

leading to

$$\tau_{R\to r_1} = 2\left(\frac{R^3}{2m}\right)^{1/2}\int_{\alpha_1}^{\pi/2} d\alpha\ \sin^2\alpha = \left(\frac{R^3}{2m}\right)^{1/2}\left[\alpha - \tfrac{1}{2}\sin 2\alpha\right]_{\alpha_1}^{\pi/2} \quad . \tag{8}$$

For $r_1 \to 0 \Leftrightarrow \alpha_1 \to 0$ one obtains

$$\tau_{R\to 0} = \left(\frac{R^3}{2m}\right)^{1/2}(\pi/2) = \pi\left(\frac{R^3}{8m}\right)^{1/2} \tag{9}$$

R and $r_S = 2m$ have dimensions of length, thus the quantity above also has dimensions of length, so what we have actually calculated is $c\tau$, not τ. To obtain proper time, we thus need to divide by c. Using the approximate values

$$(R)_{\text{sun}} \approx 7\times 10^{10}\mathsf{cm} \quad (2m)_{\text{sun}} \approx 3\times 10^{5}\mathsf{cm} \quad c \approx 3\times 10^{10}\mathsf{cm\ s}^{-1} \tag{10}$$

one finds $\tau_{R\to 0} \approx 2\times 10^3\mathsf{s}$, which is roughly 30 minutes.

2. Kruskal Coordinates for the Schwarzschild Space-Time: Solution I (direct calculation using the coordinate transformation)

To compute the Schwarzschild metric in the new (X,T)-coordinates, it is useful to consider the two expression $t(X,T)$ and $r^*(X,T)$. To find these we first rewrite (X,T) as

$$X = \mathrm{e}^{r^*/4m}\cosh(t/4m) \quad , \quad T = \mathrm{e}^{r^*/4m}\sinh(t/4m) \quad . \tag{11}$$

This leads in particular to

$$X^2 - T^2 = \mathrm{e}^{r^*/2m} = \mathrm{e}^{r/2m}\left(\frac{r}{2m} - 1\right) = rf(r)\frac{\mathrm{e}^{r/2m}}{2m} \tag{12}$$

which is a way to express r implicitly ($f(r) = (\partial r/\partial r*) = 1 - 2m/r$). Now, from (11) it also follows that

$$t = 4m\,\mathrm{atanh}\,(T/X) \quad , \quad r^* = 2m\log\left(X^2 - T^2\right) \quad . \tag{13}$$

This allows us to compute the partial derivative we will need:

$$\begin{aligned} \frac{\partial t}{\partial T} &= \frac{4mX}{X^2 - T^2} \qquad & \frac{\partial r}{\partial T} &= \frac{\partial r}{\partial r*}\frac{\partial r*}{\partial T} = F\frac{4mT}{T^2 - X^2} \\ \frac{\partial t}{\partial X} &= \frac{4mT}{T^2 - X^2} \qquad & \frac{\partial r}{\partial X} &= \frac{\partial r}{\partial r*}\frac{\partial r*}{\partial X} = F\frac{4mX}{X^2 - T^2} \end{aligned} \tag{14}$$

Then it is straightforward to compute the Schwarzschild metric starting from the old (t, r)-coordinate and we get

$$\begin{aligned} ds^2 &= -f dt^2 + f^{-1} dr^2 + r^2 d\Omega^2 \\ &= -f\left(\frac{\partial t}{\partial T} dT + \frac{\partial t}{\partial X} dX\right)^2 + f^{-1}\left(\frac{\partial r}{\partial T} dT + \frac{\partial r}{\partial X} dX\right)^2 + r^2 d\Omega^2 \\ &= \frac{16m^2 f}{(X^2 - T^2)^2}\left[-(X dT - T dX)^2 + (-T dT + X dX)^2\right] + r^2 d\Omega^2 \\ &= \frac{16m^2 f}{(X^2 - T^2)}\left[-dT^2 + dX^2\right] + r^2 d\Omega^2 \\ &= \frac{32m^3}{r} \mathrm{e}^{-r/2m}\left[-dT^2 + dX^2\right] + r^2 d\Omega^2 \end{aligned} \tag{15}$$

where in the last step we have used (12).

2. Kruskal Coordinates for the Schwarzschild Space-Time: Solution II (massaging the metric into a convenient form)

   The previous derivation may make you wonder how on earth one came up with a coordinate transformation like (11) in the first place. Here is a pedestrian way towards guessing that this might be a good transformation:

   Write the Schwarzschild metric as

$$ds^2 = (1 - 2m/r)[-dt^2 + dr^{*2}] + r^2 d\Omega^2 = (1 - 2m/r)[-du\, dv] + r(u,v)^2 d\Omega^2 \tag{16}$$

   where $r^* = r + 2m \log(r/2m - 1)$ is the tortoise coordinate, and $v = t + r^*$, $u = t - r^*$ are the "advanced" and "retarded" Eddington-Finkelstein coordinates. Now note that

$$\frac{v-u}{4m} = \frac{r}{2m} + \log\left(\frac{r}{2m} - 1\right) \quad , \tag{17}$$

   so that

$$1 - \frac{2m}{r} = \frac{2m}{r}\left(\frac{r}{2m} - 1\right) = \frac{2m}{r} \mathrm{e}^{-r/2m} \mathrm{e}^{(v-u)/4m} \quad . \tag{18}$$

   Thus the metric is

$$ds^2 = \frac{2m}{r} \mathrm{e}^{-r/2m}\left(\mathrm{e}^{v/4m} dv\right)\left(-\mathrm{e}^{-u/4m} du\right) + r(u,v)^2 d\Omega^2 \quad . \tag{19}$$

   Therefore it is natural to introduce

$$V = \mathrm{e}^{v/4m} \quad , \quad U = -\mathrm{e}^{-u/4m} \quad , \tag{20}$$

   and T and X via $V = T + X, U = T - X$, so that

$$\begin{aligned} ds^2 &= -\frac{32m^3}{r} \mathrm{e}^{-r/2m} dU dV + r(u,v)^2 d\Omega^2 \\ &= \frac{32m^3}{r} \mathrm{e}^{-r/2m}[-dT^2 + dX^2] + r(T,X)^2 d\Omega^2 \quad . \end{aligned} \tag{21}$$