Solutions to Assignments 05

1. Tensor Analysis II: the Covariant Derivative

   (a) Consider the scalar $A_\nu V^\nu$ and take its covariant derivative. Since it is a scalar, its covariant and partial derivatives agree, and since both satisfy the Leibniz rule one has

$$\begin{aligned}\nabla_\mu(A_\nu V^\nu) &= \partial_\mu(A_\nu V^\nu) = A_\nu \partial_\mu V^\nu + V^\nu \partial_\mu A_\nu \\ &= A_\nu \nabla_\mu V^\nu + V^\nu \nabla_\mu A_\nu\end{aligned} \tag{1}$$

   This implies

$$\begin{aligned}V^\nu \nabla_\mu A_\nu &= V^\nu \partial_\mu A_\nu + A_\nu \partial_\mu V^\nu - A_\nu \nabla_\mu V^\nu \\ &= V^\nu \partial_\mu A_\nu + A_\nu \partial_\mu V^\nu - A_\nu(\partial_\mu V^\nu + \Gamma^\nu_{\mu\rho} V^\rho) \\ &= V^\nu \partial_\mu A_\nu - A_\nu \Gamma^\nu_{\mu\rho} V^\rho = V^\nu \partial_\mu A_\nu - A_\lambda \Gamma^\lambda_{\mu\nu} V^\nu \\ \Rightarrow \qquad \nabla_\mu A_\nu &= \partial_\mu A_\nu - \Gamma^\lambda_{\mu\nu} A_\lambda\end{aligned} \tag{2}$$

   the last implication following because this has to be true for any V^ν.

   (b) Since $A_{\nu'} = J^\nu_{\nu'} A_\nu$ and $\partial_{\mu'} = J^\mu_{\mu'} \partial_\mu$, one has

$$\begin{aligned}\partial_\mu A_\nu \to \partial_{\mu'} A_{\nu'} &= J^\mu_{\mu'} \partial_\mu (J^\nu_{\nu'} A_\nu) \\ &= J^\mu_{\mu'} J^\nu_{\nu'} \partial_\mu A_\nu + A_\nu J^\mu_{\mu'} \partial_\mu J^\nu_{\nu'} \\ &= J^\mu_{\mu'} J^\nu_{\nu'} \partial_\mu A_\nu + A_\nu J^\nu_{\mu'\nu'} \quad .\end{aligned} \tag{3}$$

   Thus this is not a tensor, but since the last term is symmetric in the free indices,

$$J^\nu_{\mu'\nu'} = \frac{\partial^2 x^\nu}{\partial y^{\mu'} \partial y^{\nu'}} = J^\nu_{\nu'\mu'} \tag{4}$$

   (partial derivatives commute), it drops out when one takes the antisymmetric part, i.e. the curl,

$$\partial_\mu A_\nu - \partial_\nu A_\mu \to \partial_{\mu'} A_{\nu'} - \partial_{\nu'} A_{\mu'} = J^\mu_{\mu'} J^\nu_{\nu'} (\partial_\mu A_\nu - \partial_\nu A_\mu) \tag{5}$$

   Because the Christoffel symbols are symmetric in their lower indices, they always drop out of the anti-symmetrised derivatives of anti-symmetric covariant tensors. In the present (simplest) case of covectors, one has

$$\nabla_\mu A_\nu - \nabla_\nu A_\mu = \partial_\mu A_\nu - \Gamma^\lambda_{\mu\nu} A_\lambda - \partial_\nu A_\mu + \Gamma^\lambda_{\nu\mu} A_\lambda = \partial_\mu A_\nu - \partial_\nu A_\mu \quad . \tag{6}$$

   (c)
   - Argument by direct calculation: see lecture notes section 4.4.
   - Alternative argument: Since $\nabla_\mu g_{\nu\lambda}$ is a tensor, we can choose any coordinate system we like to establish if this tensor is zero or not at a given point x. Choose an inertial coordinate system at x. Then the partial derivatives of the metric and the Christoffel symbols are zero there. Therefore the covariant derivative of the metric is zero. Since $\nabla_\mu g_{\nu\lambda}$ is a tensor, this is then true in every coordinate system.

2. Tensor Analysis III: The Covariant Divergence

(a) First we compute $\Gamma^{\mu}_{\mu\lambda}$ with the definition :

$$\begin{aligned}\Gamma^{\mu}_{\mu\lambda} &= \frac{1}{2}g^{\mu\rho}(\partial_\mu g_{\rho\lambda} + \partial_\lambda g_{\mu\rho} - \partial_\rho g_{\mu\lambda}) \\ &= \frac{1}{2}(\partial^\rho g_{\rho\lambda} + g^{\mu\rho}\partial_\lambda g_{\mu\rho} - \partial^\mu g_{\mu\lambda}) \\ &= \frac{1}{2}g^{\mu\rho}\partial_\lambda g_{\mu\rho} \end{aligned} \tag{7}$$

Then, we use the relation $g^{-1}\partial_\lambda g = g^{\mu\nu}\partial_\lambda g_{\mu\nu}$ to find that :

$$\begin{aligned}\Gamma^{\mu}_{\mu\lambda} &= \frac{1}{2}g^{\mu\rho}\partial_\lambda g_{\mu\rho} \\ &= \frac{1}{2}g^{-1}\partial_\lambda g \\ &= g^{-1/2}\partial_\lambda g^{+1/2} \end{aligned} \tag{8}$$

where in the last equality we used the fact that : $\partial_\lambda g^{+1/2} = \frac{1}{2}g^{-1/2}\partial_\lambda g$.

(b) We can now compute the covariant divergence :

$$\begin{aligned}\nabla_\mu J^\mu &= \partial_\mu J^\mu + \Gamma^{\mu}_{\mu\rho}J^\rho \\ &= \partial_\mu J^\mu + J^\rho g^{-1/2}\partial_\rho g^{+1/2} \\ &= g^{-1/2}\partial_\mu(g^{1/2}J^\mu) \end{aligned} \tag{9}$$

and

$$\begin{aligned}\nabla_\mu F^{\mu\nu} &= \partial_\mu F^{\mu\nu} + \Gamma^{\mu}_{\mu\rho}F^{\rho\nu} + \Gamma^{\nu}_{\mu\rho}F^{\mu\rho} \\ &= \partial_\mu F^{\mu\nu} + \Gamma^{\mu}_{\mu\rho}F^{\rho\nu} \\ &= \partial_\mu F^{\mu\nu} + F^{\rho\nu}g^{-1/2}\partial_\rho g^{+1/2} \\ &= g^{-1/2}\partial_\mu(g^{1/2}F^{\mu\nu}) \end{aligned} \tag{10}$$

where the last term in the first equation vanishes because an antisymmetric tensor ($F^{\mu\rho}$) is contracted with a symmetric object ($\Gamma^{\nu}_{\mu\rho}$). More precisely, if we rewrite $\Gamma^{\nu}_{\mu\rho} = \frac{1}{2}(\Gamma^{\nu}_{\mu\rho} + \Gamma^{\nu}_{\rho\mu})$ and $F^{\mu\rho} = \frac{1}{2}(F^{\mu\rho} - F^{\rho\mu})$, then $\Gamma^{\nu}_{\mu\rho}F^{\mu\rho}$ contains 4 terms and relabelling two of them by the exchange of the indices $\mu \leftrightarrow \rho$ we see that everything vanishes.

(c) To calculate the Laplacian, we just need the metric,

$$ds^2 = dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \quad \Leftrightarrow \quad (g_{\alpha\beta}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{pmatrix} \tag{11}$$

its inverse,

$$(g^{\alpha\beta}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^{-2} & 0 \\ 0 & 0 & r^{-2}(\sin\theta)^{-2} \end{pmatrix} \tag{12}$$

and its determinant,

$$g = r^4 \sin^2\theta \quad \Rightarrow \quad \sqrt{g} = r^2 \sin\theta \tag{13}$$

Then one calculates

$$\begin{aligned}\Box\Phi &= \frac{1}{r^2\sin\theta}\partial_\alpha(r^2\sin\theta g^{\alpha\beta}\partial_\beta\Phi) \\ &= \frac{1}{r^2\sin\theta}\left(\partial_r(r^2\sin\theta\partial_r\Phi) + \partial_\theta(\sin\theta\partial_\theta\Phi) + \partial_\phi((\sin\theta)^{-1}\partial_\theta\Phi)\right) \\ &= r^{-2}\partial_r(r^2\partial_r\Phi) + r^{-2}\left((\sin\theta)^{-1}\partial_\theta(\sin\theta\partial_\theta\Phi) + (\sin\theta)^{-2}\partial_\phi^2\Phi\right)\end{aligned} \tag{14}$$

This can now be rewritten in many ways, e.g. as

$$\Box = \partial_r^2 + \frac{2}{r}\partial_r + \frac{\Delta_{S^2}}{r^2} \tag{15}$$

with

$$\Delta_{S^2} = \frac{1}{\sin\theta}\partial_a(\sin\theta g^{ab}\partial_b) \tag{16}$$

$(x^a = (\theta,\phi))$ the Laplace operator on the unit 2-sphere.