1. Tensor Analysis II: the Covariant Derivative

(a) Consider the scalar $A_\nu V^\nu$ and take its covariant derivative. Since it is a scalar, its covariant and partial derivatives agree, and since both satisfy the Leibniz rule one has

$$\nabla_\mu (A_\nu V^\nu) = \partial_\mu (A_\nu V^\nu) = A_\nu \partial_\mu V^\nu + V^\nu \partial_\mu A_\nu$$

(1)

This implies

$$V^\nu \nabla_\mu A_\nu = \partial_\mu A_\nu - \Gamma^\lambda_\mu\nu A_\lambda$$

(2)

the last implication following because this has to be true for any $V^\nu$.

(b) Since $A_\nu' = J^\nu_\mu A_\nu$ and $\partial_\mu' = J^\mu_\nu \partial_\mu$, one has

$$\partial_\mu A_\nu \rightarrow \partial_\mu' A_\nu' = J^\mu_\nu \partial_\mu (J^\nu_\mu A_\nu)$$

(3)

$$= J^\mu_\nu J^\nu_\mu \partial_\mu A_\nu + A_\nu J^\mu_\nu \partial_\mu J^\nu_\mu$$

Thus this is not a tensor, but since the last term is symmetric in the free indices,

$$J^\nu_\mu J^\lambda_\nu = \frac{\partial^2 x^\nu}{\partial y^\mu \partial y^\nu} = J^\nu_\mu J^\lambda_\nu$$

(4)

(partial derivatives commute), it drops out when one takes the antisymmetric part, i.e. the curl,

$$\partial_\mu A_\nu - \partial_\nu A_\mu \rightarrow \partial_\mu' A_\nu' - \partial_\nu' A_\mu' = J^\mu_\nu J^\nu_\mu (\partial_\mu A_\nu - \partial_\nu A_\mu)$$

(5)

Because the Christoffel symbols are symmetric in their lower indices, they always drop out of the anti-symmetrised derivatives of anti-symmetric covariant tensors. In the present (simplest) case of covectors, one has

$$\nabla_\mu A_\nu - \nabla_\nu A_\mu = \partial_\mu A_\nu - \Gamma^\lambda_\mu\nu A_\lambda - \partial_\nu A_\mu + \Gamma^\lambda_\nu\mu A_\lambda = \partial_\mu A_\nu - \partial_\nu A_\mu$$

(6)

(c) • Argument by direct calculation: see lecture notes section 4.4.

• Alternative argument: Since $\nabla_\mu g_{\nu\lambda}$ is a tensor, we can choose any coordinate system we like to establish if this tensor is zero or not at a given point $x$. Choose an inertial coordinate system at $x$. Then the partial derivatives of the metric and the Christoffel symbols are zero there. Therefore the covariant derivative of the metric is zero. Since $\nabla_\mu g_{\nu\lambda}$ is a tensor, this is then true in every coordinate system.
2. Tensor Analysis III: The Covariant Divergence

(a) First we compute $\Gamma^\mu_{\mu\lambda}$ with the definition:

$$\Gamma^\mu_{\mu\lambda} = \frac{1}{2} g^{\mu\rho} \left( \partial_\mu g_{\rho\lambda} + \partial_\lambda g_{\mu\rho} - \partial_\rho g_{\mu\lambda} \right)$$

$$= \frac{1}{2} \left( \partial_\rho g_{\rho\lambda} + g^{\mu\rho} \partial_\lambda g_{\mu\rho} - \partial_\mu g_{\mu\lambda} \right)$$

$$= \frac{1}{2} g^{\mu\rho} \partial_\lambda g_{\mu\rho} \quad (7)$$

Then, we use the relation $g^{-1} \partial_\lambda g = g^{\mu\nu} \partial_\lambda g_{\mu\nu}$ to find that:

$$\Gamma^\mu_{\mu\lambda} = \frac{1}{2} g^{\mu\rho} \partial_\lambda g_{\mu\rho}$$

$$= g^{-1/2} \partial_\lambda g$$

$$= g^{-1/2} g^{\mu\nu} \partial_\lambda g_{\mu\nu} + \frac{1}{2} \partial_\lambda g \quad (8)$$

where in the last equality we used the fact that $\partial_\lambda g + \frac{1}{2} \partial_\lambda g = \frac{1}{2} g^{-1/2} \partial_\lambda g + \frac{1}{2} \partial_\lambda g$.

(b) We can now compute the covariant divergence:

$$\nabla^\mu J^\mu = \partial_\mu J^\mu + \Gamma^\mu_{\mu\rho} J^\rho$$

$$= \partial_\mu J^\mu + J^\rho g^{-1/2} \partial_\rho g^{1/2}$$

$$= g^{-1/2} \partial_\mu (g^{1/2} J^\mu) \quad (9)$$

and

$$\nabla^\mu F^{\mu\nu} = \partial_\mu F^{\mu\nu} + \Gamma^\mu_{\mu\rho} F^{\rho\nu} + \Gamma^\nu_{\mu\rho} F^{\mu\rho}$$

$$= \partial_\mu F^{\mu\nu} + \Gamma^\mu_{\mu\rho} F^{\rho\nu}$$

$$= \partial_\mu F^{\mu\nu} + F^{\rho\nu} g^{-1/2} \partial_\rho g^{1/2}$$

$$= g^{-1/2} \partial_\mu (g^{1/2} F^{\mu\nu}) \quad (10)$$

where the last term in the first equation vanishes because an antisymmetric tensor ($F^{\mu\nu}$) is contracted with a symmetric object ($\Gamma^\nu_{\mu\rho}$). More precisely, if we rewrite $\Gamma^\nu_{\mu\rho} = \frac{1}{2} (\Gamma^\nu_{\nu\rho} + \Gamma^\nu_{\rho\nu})$ and $F^{\mu\nu} = \frac{1}{2} (F^{\nu\mu} - F^{\mu\nu})$, then $\Gamma^\nu_{\mu\rho} F^{\mu\nu}$ contains 4 terms and relabelling two of them by the exchange of the indices $\mu \leftrightarrow \rho$ we see that everything vanishes.

(c) To calculate the Laplacian, we just need the metric,

$$ds^2 = dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \quad \Leftrightarrow \quad (g_{\alpha\beta}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2 \theta \end{pmatrix} \quad (11)$$

its inverse,

$$(g^{\alpha\beta}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^{-2} & 0 \\ 0 & 0 & r^{-2} (\sin \theta)^{-2} \end{pmatrix} \quad (12)$$
and its determinant,

\[ g = r^4 \sin^2 \theta \quad \Rightarrow \quad \sqrt{g} = r^2 \sin \theta \quad (13) \]

Then one calculates

\[
\Box \Phi = \frac{1}{r^2 \sin \theta} \partial_\alpha (r^2 \sin \theta g^{\alpha \beta} \partial_\beta \Phi) \\
= \frac{1}{r^2 \sin \theta} \left( \partial_r (r^2 \sin \theta \partial_r \Phi) + \partial_\theta (\sin \theta \partial_\theta \Phi) + \partial_\phi ((\sin \theta)^{-1} \partial_\theta \Phi) \right) \\
= r^{-2} \partial_r (r^2 \partial_r \Phi) + r^{-2} \left( (\sin \theta)^{-1} \partial_\theta (\sin \theta \partial_\theta \Phi) + (\sin \theta)^{-2} \partial_\phi^2 \Phi \right) \quad (14)
\]

This can now be rewritten in many ways, e.g. as

\[
\Box = \partial_r^2 + \frac{2}{r} \partial_r + \frac{\Delta_{S^2}}{r^2} \quad (15)
\]

with

\[
\Delta_{S^2} = \frac{1}{\sin \theta} \partial_\alpha (\sin \theta g^{\alpha \beta} \partial_\beta) \quad (16)
\]

\((x^a = (\theta, \phi))\) the Laplace operator on the unit 2-sphere.