# Solutions to Assignments 06

1. Properties of the Riemann Curvature Tensor

   (a) We show that the fourth symmetry follows from (I),(II) and (III):

$$\begin{aligned} R_{\alpha\beta\gamma\delta} &= -(R_{\alpha\delta\beta\gamma} + R_{\alpha\gamma\delta\beta}) = R_{\gamma\alpha\delta\beta} + R_{\delta\alpha\beta\gamma} \\ &= -(R_{\gamma\delta\beta\alpha} + R_{\gamma\beta\alpha\delta}) - (R_{\delta\beta\gamma\alpha} + R_{\delta\gamma\alpha\beta}) \\ &= 2R_{\gamma\delta\alpha\beta} + R_{\beta\gamma\alpha\delta} + R_{\beta\delta\gamma\alpha} \\ &= 2R_{\gamma\delta\alpha\beta} - R_{\beta\alpha\delta\gamma} = 2R_{\gamma\delta\alpha\beta} - R_{\alpha\beta\gamma\delta} \\ \Rightarrow&\ \ R_{\alpha\beta\gamma\delta} = R_{\gamma\delta\alpha\beta} \end{aligned} \tag{1}$$

   (b) From (a) we directly deduce the symmetry of the Ricci tensor :

$$R_{\mu\nu} = R^{\rho}{}_{\mu\rho\nu} = R_{\rho\nu}{}^{\rho}{}_{\mu} = R^{\rho}{}_{\nu\rho\mu} = R_{\nu\mu} \tag{2}$$

   (c) Writing $\circlearrowleft$ for the cyclic permutations in $(\alpha, \beta, \gamma)$ and then using the third symmetry : $R^{\rho}{}_{\alpha\beta\gamma} + \circlearrowleft = 0$, we have :

$$\begin{aligned} [\nabla_\alpha, [\nabla_\beta, \nabla_\gamma]]V^\lambda + \circlearrowleft \quad &= \nabla_\alpha(R^{\lambda}{}_{\rho\beta\gamma}V^\rho) - R^{\lambda}{}_{\rho\beta\gamma}\nabla_\alpha V^\rho + R^{\rho}{}_{\alpha\beta\gamma}\nabla_\rho V^\lambda + \circlearrowleft \\ &= \nabla_\alpha(R^{\lambda}{}_{\rho\beta\gamma})V^\rho + R^{\rho}{}_{\alpha\beta\gamma}\nabla_\rho V^\lambda + \circlearrowleft \\ &= \nabla_\alpha(R^{\lambda}{}_{\rho\beta\gamma})V^\rho + \circlearrowleft \\ &= g^{\lambda\mu}\left[\nabla_\alpha R_{\mu\nu\beta\gamma} + \circlearrowleft\right] V^\nu = 0 \end{aligned} \tag{3}$$

   which gives the desired result.

   (d) Contracting the Bianchi identity over the indices $(\mu, \beta)$ and $(\nu, \alpha)$ one finds :

$$\begin{aligned} g^{\nu\alpha}g^{\mu\beta}\left[\nabla_\alpha R_{\mu\nu\beta\gamma} + \circlearrowleft\right] \quad &= g^{\nu\alpha}g^{\mu\beta}\left[\nabla_\alpha R_{\mu\nu\beta\gamma} + \nabla_\beta R_{\mu\nu\gamma\alpha} + \nabla_\gamma R_{\mu\nu\alpha\beta}\right] \\ &= \nabla_\alpha R^{\alpha\beta}{}_{\beta\gamma} + \nabla_\beta R^{\alpha\beta}{}_{\gamma\alpha} + \nabla_\gamma R^{\alpha\beta}{}_{\alpha\beta} \\ &= -\nabla_\alpha R^{\alpha}{}_{\gamma} - \nabla_\beta R^{\beta}{}_{\gamma} + \nabla_\gamma R \\ &= -\nabla_\alpha\left[2R^{\alpha}{}_{\gamma} - \delta^{\alpha}_{\gamma}R\right] = 0 \end{aligned} \tag{4}$$

   And defining the *Einstein tensor* as $G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R$, we see that the contracted Bianchi identity (4) is equivalent to $\nabla^\alpha G_{\alpha\beta} = 0$ because :

$$\nabla^\alpha G_{\alpha\beta} = \nabla^\alpha(R_{\alpha\beta} - \tfrac{1}{2}g_{\alpha\beta}R) = \frac{1}{2}\nabla_\alpha(2R^{\alpha}{}_{\beta} - g^{\alpha}{}_{\beta}R) \tag{5}$$

   so that $\nabla^\alpha G_{\alpha\beta} = 0 \;\Leftrightarrow\; \nabla_\alpha\left[2R^{\alpha}{}_{\gamma} - \delta^{\alpha}_{\gamma}R\right] = 0$ where we have use the fact that $g^{\alpha}{}_{\beta} = g^{\alpha\lambda}g_{\lambda\beta} = \delta^{\alpha}{}_{\beta}$ simply because $g^{\alpha\lambda}$ is the inverse of $g_{\alpha\lambda}$.

2. Curvature in 2 Dimensions

(a) Since the Ricci tensor is

$$R_{\alpha\beta} = R^{\gamma}{}_{\alpha\gamma\beta} = R^{1}{}_{\alpha 1\beta} + R^{2}{}_{\alpha 2\beta} \tag{6}$$

because of the anti-symmetry of the Riemann tensor in the last two indices one has $R_{11} = R^{2}{}_{121}$ etc. Then the scalar curvature is

$$R = g^{\alpha\beta}R_{\alpha\beta} = g^{11}R^{2}{}_{121} + g^{12}R^{1}{}_{112} + g^{21}R^{2}{}_{221} + g^{22}R^{1}{}_{212} \quad . \tag{7}$$

Using

$$\left(g^{\alpha\beta}\right) = \frac{1}{g_{11}g_{22} - g_{12}g_{21}}\begin{pmatrix} g_{22} & -g_{12} \\ -g_{21} & g_{11} \end{pmatrix} \tag{8}$$

and

$$R_{1212} = g_{1\alpha}R^{\alpha}{}_{212} = g_{11}R^{1}{}_{212} + g_{12}R^{2}{}_{212} \tag{9}$$

(and likewise for $R_{2112} = -R_{2121}$) one then finds

$$R = \frac{2}{g_{11}g_{22} - g_{12}g_{21}}R_{1212} \quad . \tag{10}$$

(b) From the Euler-Lagrange equations associated to the metric $ds^2 = dx^2 + e^{2x}dy^2$ one reads off the non-zero Christoffel symbols

$$\ddot{x} - \mathrm{e}^{2x}\dot{y}^2 = 0 \quad \Rightarrow \quad \Gamma^{x}_{yy} = -\mathrm{e}^{2x} \qquad \ddot{y} + 2\dot{x}\dot{y} = 0 \quad \Rightarrow \quad \Gamma^{y}_{xy} = 1 \quad . \tag{11}$$

Then

$$\begin{aligned} R_{xyxy} = g_{x\alpha}R^{\alpha}{}_{yxy} = R^{x}{}_{yxy} &= \partial_x\Gamma^{x}_{yy} - \partial_y\Gamma^{x}_{yx} + \Gamma^{x}_{x\alpha}\Gamma^{\alpha}_{yy} - \Gamma^{x}_{y\alpha}\Gamma^{\alpha}_{yx} \\ &= \partial_x\Gamma^{x}_{yy} - \Gamma^{x}_{yy}\Gamma^{y}_{yx} = -\mathrm{e}^{2x} \end{aligned} \tag{12}$$

and thus $R = 2g^{-1}(-\mathrm{e}^{2x}) = -2$.

3. The Geodesic Deviation Equation (section 8.3)

(a) This is obvious (I hope) since from expansion of the deplaced equation to first order one gets

$$\begin{aligned} &\Gamma^{\mu}{}_{\nu\lambda}(x+\delta x)\tfrac{d}{d\tau}(x^{\nu} + \delta x^{\nu})\tfrac{d}{d\tau}(x^{\lambda} + \delta x^{\lambda}) \\ &\qquad = \partial_{\rho}\Gamma^{\mu}{}_{\nu\lambda}(x)\delta x^{\rho}\tfrac{d}{d\tau}x^{\nu}\tfrac{d}{d\tau}x^{\lambda} + 2\Gamma^{\mu}{}_{\nu\lambda}(x)\tfrac{d}{d\tau}x^{\nu}\tfrac{d}{d\tau}\delta x^{\lambda} \end{aligned} \tag{13}$$

(the symmetry of the Christoffel symbols accounting for the factor of 2).

(b) Starting from (here I write $D/D\tau$ instead of $D_{\tau}$)

$$\frac{D}{D\tau}\delta x^{\mu} = \frac{d}{d\tau}\delta x^{\mu} + \Gamma^{\mu}{}_{\nu\lambda}\frac{dx^{\nu}}{d\tau}\delta x^{\lambda} \tag{14}$$

one can calculate the 2nd derivative $D^2\delta x^\mu/D\tau^2$,

$$\begin{aligned}\frac{D^2}{D\tau^2}\dot{x}^\mu &= \frac{d}{d\tau}\left[\delta\dot{x}^\mu + \Gamma^\mu_{\ \nu\rho}\dot{x}^\nu\delta x^\rho\right] + \Gamma^\mu_{\ \alpha\beta}\dot{x}^\alpha\left[\delta\dot{x}^\beta + \Gamma^\beta_{\ \nu\rho}\dot{x}^\nu\delta x^\rho\right] \\ &= \delta\ddot{x}^\mu + (\partial_\lambda\Gamma^\mu_{\ \nu\rho})\dot{x}^\nu\dot{x}^\lambda\delta x^\rho + \Gamma^\mu_{\ \nu\rho}\ddot{x}^\nu\delta x^\rho + 2\Gamma^\mu_{\ \nu\rho}\dot{x}^\nu\delta\dot{x}^\rho \\ &+ \Gamma^\mu_{\ \lambda\beta}\Gamma^\beta_{\ \nu\rho}\dot{x}^\lambda\dot{x}^\nu\delta x^\rho\end{aligned} \tag{15}$$

Subtracting from this the term $R^\mu_{\ \nu\lambda\rho}\dot{x}^\nu\dot{x}^\lambda\delta x^\rho$ and using the geodesic equation to eliminate $\ddot{x}^\nu$, one finds the equation

$$\tfrac{d^2}{d\tau^2}\delta x^\mu + 2\Gamma^\mu_{\ \nu\lambda}(x)\tfrac{d}{d\tau}x^\nu\tfrac{d}{d\tau}\delta x^\lambda + \partial_\rho\Gamma^\mu_{\ \nu\lambda}(x)\delta x^\rho\tfrac{d}{d\tau}x^\nu\tfrac{d}{d\tau}x^\lambda = 0 \tag{16}$$