Solutions to Assignments 05

1. Tensor Analysis III: The Covariant Divergence

   (a) First we compute $\Gamma^{\mu}_{\mu\lambda}$ with the definition :

$$\begin{aligned} \Gamma^{\mu}_{\mu\lambda} &= \frac{1}{2} g^{\mu\rho}(\partial_\mu g_{\rho\lambda} + \partial_\lambda g_{\mu\rho} - \partial_\rho g_{\mu\lambda}) \\ &= \frac{1}{2}(\partial^\rho g_{\rho\lambda} + g^{\mu\rho}\partial_\lambda g_{\mu\rho} - \partial^\mu g_{\mu\lambda}) \\ &= \frac{1}{2} g^{\mu\rho}\partial_\lambda g_{\mu\rho} \end{aligned} \tag{1}$$

   Then, we use the relation $g^{-1}\partial_\lambda g = g^{\mu\nu}\partial_\lambda g_{\mu\nu}$ to find that :

$$\begin{aligned} \Gamma^{\mu}_{\mu\lambda} &= \frac{1}{2} g^{\mu\rho}\partial_\lambda g_{\mu\rho} \\ &= \frac{1}{2} g^{-1}\partial_\lambda g \\ &= g^{-1/2}\partial_\lambda g^{+1/2} \end{aligned} \tag{2}$$

   where in the last equality we used the fact that : $\partial_\lambda g^{+1/2} = \frac{1}{2} g^{-1/2}\partial_\lambda g$.

   (b) We can now compute the covariant divergence :

$$\begin{aligned} \nabla_\mu J^\mu &= \partial_\mu J^\mu + \Gamma^{\mu}_{\mu\rho} J^\rho \\ &= \partial_\mu J^\mu + J^\rho g^{-1/2}\partial_\rho g^{+1/2} \\ &= g^{-1/2}\partial_\mu (g^{1/2} J^\mu) \end{aligned} \tag{3}$$

   and

$$\begin{aligned} \nabla_\mu F^{\mu\nu} &= \partial_\mu F^{\mu\nu} + \Gamma^{\mu}_{\mu\rho} F^{\rho\nu} + \Gamma^{\nu}_{\mu\rho} F^{\mu\rho} \\ &= \partial_\mu F^{\mu\nu} + \Gamma^{\mu}_{\mu\rho} F^{\rho\nu} \\ &= \partial_\mu F^{\mu\nu} + F^{\rho\nu} g^{-1/2}\partial_\rho g^{+1/2} \\ &= g^{-1/2}\partial_\mu (g^{1/2} F^{\mu\nu}) \end{aligned} \tag{4}$$

   where the last term in the first equation vanishes because an antisymmetric tensor ($F^{\mu\rho}$) is contracted with a symmetric object ($\Gamma^{\nu}_{\mu\rho}$). More precisely, if we rewrite $\Gamma^{\nu}_{\mu\rho} = \frac{1}{2}(\Gamma^{\nu}_{\mu\rho} + \Gamma^{\nu}_{\rho\mu})$ and $F^{\mu\rho} = \frac{1}{2}(F^{\mu\rho} - F^{\rho\mu})$, then $\Gamma^{\nu}_{\mu\rho} F^{\mu\rho}$ contains 4 terms and relabelling two of them by the exchange of the indices $\mu \leftrightarrow \rho$ we see that everything vanishes.

   (c) To calculate the Laplacian, we just need the metric,

$$ds^2 = dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \quad \Leftrightarrow \quad (g_{\alpha\beta}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{pmatrix} \tag{5}$$

its inverse,

$$(g^{\alpha\beta}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^{-2} & 0 \\ 0 & 0 & r^{-2}(\sin\theta)^{-2} \end{pmatrix} \tag{6}$$

and its determinant,

$$g = r^4 \sin^2\theta \quad \Rightarrow \quad \sqrt{g} = r^2 \sin\theta \tag{7}$$

Then one calculates

$$\begin{aligned} \Box\Phi &= \frac{1}{r^2\sin\theta}\partial_\alpha(r^2\sin\theta g^{\alpha\beta}\partial_\beta\Phi) \\ &= \frac{1}{r^2\sin\theta}\left(\partial_r(r^2\sin\theta\partial_r\Phi) + \partial_\theta(\sin\theta\partial_\theta\Phi) + \partial_\phi((\sin\theta)^{-1}\partial_\theta\Phi)\right) \\ &= r^{-2}\partial_r(r^2\partial_r\Phi) + r^{-2}\left((\sin\theta)^{-1}\partial_\theta(\sin\theta\partial_\theta\Phi) + (\sin\theta)^{-2}\partial_\phi^2\Phi\right) \end{aligned} \tag{8}$$

This can now be rewritten in many ways, e.g. as

$$\Box = \partial_r^2 + \frac{2}{r}\partial_r + \frac{\Delta_{S^2}}{r^2} \tag{9}$$

with

$$\Delta_{S^2} = \frac{1}{\sin\theta}\partial_a(\sin\theta g^{ab}\partial_b) \tag{10}$$

($x^a = (\theta,\phi)$) the Laplace operator on the unit 2-sphere.

Remark: This calculation evidently generalises to any dimension. Thus in spherical coordinates in which the Euclidean metric has the form

$$ds^2 = dr^2 + r^2 d\Omega_n^2 \tag{11}$$

(with $d\Omega_n^2$ the line element on the unit n-sphere), the Laplace operator on $\mathbb{R}^{n+1}$ takes the form

$$\Box = \partial_r^2 + \frac{n}{r}\partial_r + \frac{\Delta_{S^n}}{r^2} \tag{12}$$

2. Kruskal Coordinates for the Schwarzschild Space-Time: Solution I (direct calculation using the coordinate transformation)

To compute the Schwarzschild metric in the new (X,T)-coordinates, it is useful to consider the two expression $t(X,T)$ and $r^*(X,T)$. To find these we first rewrite (X,T) as

$$X = \mathrm{e}^{r^*/4m}\cosh(t/4m) \quad , \quad T = \mathrm{e}^{r^*/4m}\sinh(t/4m) \ . \tag{13}$$

This leads in particular to

$$X^2 - T^2 = \mathrm{e}^{r^*/2m} = \mathrm{e}^{r/2m}\left(\frac{r}{2m} - 1\right) = rf(r)\frac{\mathrm{e}^{r/2m}}{2m} \tag{14}$$

which is a way to express r implicitly ($f(r) = (\partial r/\partial r*) = 1 - 2m/r$). Now, from (13) it also follows that

$$t = 4m\,\mathrm{atanh}\,(T/X) \quad , \quad r^* = 2m\log\left(X^2 - T^2\right) \quad . \tag{15}$$

This allows us to compute the partial derivative we will need:

$$\begin{aligned} \frac{\partial t}{\partial T} &= \frac{4mX}{X^2 - T^2} & \qquad \frac{\partial r}{\partial T} &= \frac{\partial r}{\partial r*}\frac{\partial r*}{\partial T} = F\frac{4mT}{T^2 - X^2} \\ \frac{\partial t}{\partial X} &= \frac{4mT}{T^2 - X^2} & \qquad \frac{\partial r}{\partial X} &= \frac{\partial r}{\partial r*}\frac{\partial r*}{\partial X} = F\frac{4mX}{X^2 - T^2} \end{aligned} \tag{16}$$

Then it is straightforward to compute the Schwarzschild metric starting from the old (t, r)-coordinate and we get

$$\begin{aligned} ds^2 &= -f dt^2 + f^{-1} dr^2 + r^2 d\Omega^2 \\ &= -f\left(\frac{\partial t}{\partial T}dT + \frac{\partial t}{\partial X}dX\right)^2 + f^{-1}\left(\frac{\partial r}{\partial T}dT + \frac{\partial r}{\partial X}dX\right)^2 + r^2 d\Omega^2 \\ &= \frac{16m^2 f}{(X^2 - T^2)^2}\left[-(XdT - TdX)^2 + (-TdT + XdX)^2\right] + r^2 d\Omega^2 \\ &= \frac{16m^2 f}{(X^2 - T^2)}\left[-dT^2 + dX^2\right] + r^2 d\Omega^2 \\ &= \frac{32m^3}{r}\,\mathrm{e}^{-r/2m}\left[-dT^2 + dX^2\right] + r^2 d\Omega^2 \end{aligned} \tag{17}$$

where in the last step we have used (14).

2. Kruskal Coordinates for the Schwarzschild Space-Time: Solution II (massaging the metric into a convenient form)

The previous derivation may make you wonder how on earth one came up with a coordinate transformation like (13) in the first place. Here is a pedestrian way towards guessing that this might be a good transformation:

Write the Schwarzschild metric as

$$ds^2 = (1 - 2m/r)[-dt^2 + dr^{*2}] + r^2 d\Omega^2 = (1 - 2m/r)[-du\, dv] + r(u, v)^2 d\Omega^2 \tag{18}$$

where $r^* = r + 2m\log(r/2m - 1)$ is the tortoise coordinate, and $v = t + r^*$, $u = t - r^*$ are the "advanced" and "retarded" Eddington-Finkelstein coordinates. Now note that

$$\frac{v - u}{4m} = \frac{r}{2m} + \log\left(\frac{r}{2m} - 1\right) \quad , \tag{19}$$

so that

$$1 - \frac{2m}{r} = \frac{2m}{r}\left(\frac{r}{2m} - 1\right) = \frac{2m}{r}\mathrm{e}^{-r/2m}\mathrm{e}^{(v-u)/4m} \quad . \tag{20}$$

Thus the metric is

$$ds^2 = \frac{2m}{r}\mathrm{e}^{-r/2m}\left(\mathrm{e}^{v/4m}dv\right)\left(-\mathrm{e}^{-u/4m}du\right) + r(u, v)^2 d\Omega^2 \quad . \tag{21}$$

Therefore it is natural to introduce

$$V = \mathrm{e}^{v/4m} \quad , \quad U = -\mathrm{e}^{-u/4m} \ , \tag{22}$$

and T and X via $V = T + X, U = T - X$, so that

$$\begin{aligned} ds^2 &= -\frac{32m^3}{r}\mathrm{e}^{-r/2m} dU dV + r(u,v)^2 d\Omega^2 \\ &= \frac{32m^3}{r}\mathrm{e}^{-r/2m}[-dT^2 + dX^2] + r(T,X)^2 d\Omega^2 \ . \end{aligned} \tag{23}$$