Solutions to Assignments 01

1. Coordinate Transformations and Metrics in Minkowski Space

(a) There are (at least) 2 ways to do this calculation. The longer one is to use the formula $g_{\mu\nu} = J^a_\mu J^b_\nu \eta_{ab}$ to compute the components of the metric in the new coordinates one by one:

$$\begin{aligned} g_{TT} &= \eta_{ab}\frac{\partial\xi^a}{\partial T}\frac{\partial\xi^b}{\partial T} = -\left(\frac{\partial t}{\partial T}\right)^2 + \left(\frac{\partial x}{\partial T}\right)^2 \\ &= -X^2\cosh(T)^2 + X^2\sinh(T)^2 = -X^2 \end{aligned} \tag{1}$$

$$\begin{aligned} g_{TX} &= \eta_{ab}\frac{\partial\xi^a}{\partial T}\frac{\partial\xi^b}{\partial X} = -\frac{\partial t}{\partial T}\frac{\partial t}{\partial X} + \frac{\partial x}{\partial T}\frac{\partial x}{\partial X} \\ &= -X\cosh(T)\sinh(T) + X\sinh(T)\cosh(T) = 0 \end{aligned} \tag{2}$$

$$\begin{aligned} g_{XX} &= \eta_{ab}\frac{\partial\xi^a}{\partial X}\frac{\partial\xi^b}{\partial X} = -\left(\frac{\partial t}{\partial X}\right)^2 + \left(\frac{\partial x}{\partial X}\right)^2 \\ &= -\sinh(T)^2 + \cosh(T)^2 = 1 \end{aligned} \tag{3}$$

Thus, assembling the results, we can read off that the Minkowski line-element in Rindler coordinates takes the form

$$ds^2 = -X^2 dT^2 + dX^2 \tag{4}$$

Alternatively (and this is frequently the calculationally more efficient way of proceeding, even though in the present example it makes hardly any difference), one can simply calculate dt and dx in terms of the new coordinates once and for all, and then plug the result into the line element to read off the components of the metric:

$$\begin{aligned} t &= X\sinh T \quad , \quad x = X\cosh T \\ dt &= dX\ \sinh T + X\cosh T\ dT \quad , \quad dx = dX\ \cosh T + X\sinh T\ dT \\ ds^2 &= -(dX\ \sinh T + X\cosh T\ dT)^2 + (dX\ \cosh T + X\sinh T\ dT)^2 \\ &= -X^2 dT^2 + dX^2\ . \end{aligned} \tag{5}$$

(b) With

$$g^{\mu\nu} = \eta^{ab}J^\mu_a J^\nu_b \quad , \quad g_{\nu\lambda} = J^c_\nu J^d_\lambda \eta_{cd} \tag{6}$$

we calculate, using $J^\nu_b J^\nu_c = \delta^b_c$, $\eta^{ab}\eta_{bd} = \delta^b_d$ etc.

$$g^{\mu\nu}g_{\nu\lambda} = \eta^{ab}J^\mu_a J^\nu_b J^c_\nu J^d_\lambda \eta_{cd} = \eta^{ab}J^\mu_a J^d_\lambda \eta_{bd} = J^\mu_a J^a_\lambda = \delta^\mu_\lambda \tag{7}$$

2. Free Relativistic Particle in Arbitrary Coordinates

- From
$$g_{\mu\nu} = \eta_{ab} J^a_\mu J^b_\nu \tag{8}$$
one deduces
$$g_{\mu\nu,\lambda} = \eta_{ab}(J^a_{\mu\lambda} J^b_\nu + J^a_\mu J^b_{\nu\lambda}) \tag{9}$$
where
$$J^a_{\mu\lambda} = \partial_\lambda J^a_\mu = \frac{\partial^2 \xi^a}{\partial x^\mu \partial x^\lambda} = J^a_{\lambda\mu} \quad . \tag{10}$$
- Therefore one has
$$\begin{aligned}\Gamma_{\mu\nu\lambda} &= \tfrac{1}{2}(g_{\mu\nu,\lambda} + g_{\mu\lambda,\nu} - g_{\nu\lambda,\mu}) \\ &= \tfrac{1}{2}\eta_{ab}(J^a_{\mu\lambda}J^b_\nu + J^a_\mu J^b_{\nu\lambda} + J^a_{\mu\nu}J^b_\lambda + J^a_\mu J^b_{\lambda\nu} - J^a_{\nu\mu}J^b_\lambda - J^a_\nu J^b_{\lambda\mu}) \\ &= \eta_{ab} J^a_\mu J^b_{\nu\lambda} \quad ,\end{aligned} \tag{11}$$
where the cancellations in passing to the last line arise from the symmetries $\eta_{ab} = \eta_{ba}$, $J^b_{\lambda\mu} = J^b_{\mu\lambda}$ etc.
- Thus (writing out everything in detail),
$$\begin{aligned}\Gamma^\mu{}_{\nu\lambda} &= g^{\mu\rho}\Gamma_{\rho\nu\lambda} = \eta^{cd} J^\mu_c J^\rho_d \eta_{ab} J^a_\rho J^b_{\nu\lambda} = \eta^{cd} J^\mu_c \delta^a_d \eta_{ab} J^b_{\nu\lambda} \\ &= \eta^{ca} J^\mu_c \eta_{ab} J^b_{\nu\lambda} = \delta^c_b J^\mu_c J^b_{\nu\lambda} = J^\mu_b J^b_{\nu\lambda} \quad ,\end{aligned} \tag{12}$$
as was to be shown.

3. Geodesics

(a) With the Lagrangian
$$\mathcal{L}(x^\mu, \dot{x}^\mu) = \frac{1}{2} g_{\mu\nu} \frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = \frac{1}{2} g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu \tag{13}$$
where $g_{\mu\nu} = g_{\mu\nu}(x^\rho)$ one computes
$$\frac{\partial \mathcal{L}}{\partial x^\mu} = \frac{1}{2}\dot{x}^\rho \dot{x}^\nu \partial_\mu g_{\rho\nu} \tag{14}$$
$$\frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} = g_{\rho\nu}\dot{x}^\rho \frac{\partial}{\partial \dot{x}^\mu}\dot{x}^\nu = g_{\rho\nu}\dot{x}^\rho \delta^\nu_\mu = g_{\rho\mu}\dot{x}^\rho \tag{15}$$
$$\frac{d}{d\tau}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}^\mu}\right) = g_{\rho\mu}\ddot{x}^\rho + \dot{x}^\rho \dot{x}^\nu \partial_\nu g_{\rho\mu} = g_{\rho\mu}\ddot{x}^\rho + \frac{1}{2}\left(\dot{x}^\rho \dot{x}^\nu \partial_\nu g_{\rho\mu} + \dot{x}^\nu \dot{x}^\rho \partial_\rho g_{\nu\mu}\right) \tag{16}$$
Thus the Euler-Lagrange equations become
$$\begin{aligned}\Big[\text{E.-L.}\Big] &= g_{\mu\rho}\ddot{x}^\rho + \frac{1}{2}\left(\partial_\nu g_{\rho\mu} + \partial_\rho g_{\nu\mu} - \partial_\mu g_{\rho\nu}\right)\dot{x}^\nu \dot{x}^\rho & (17)\\ &= g_{\mu\rho}\ddot{x}^\rho + \Gamma_{\mu\nu\rho}\dot{x}^\rho \dot{x}^\nu = 0 & (18)\end{aligned}$$
and they can be written in the usual (geodesic equation) form by multiplying by $g^{\lambda\mu}$ to move the index μ up:
$$g^{\lambda\mu}\left(g_{\mu\rho}\ddot{x}^\rho + \Gamma_{\mu\nu\rho}\dot{x}^\rho \dot{x}^\nu\right) = \ddot{x}^\lambda + \Gamma^\lambda_{\nu\rho}\dot{x}^\rho \dot{x}^\nu = 0 \tag{19}$$

(b) First we compute :

$$\frac{d}{d\tau}\mathcal{L} = \frac{1}{2}\frac{d}{d\tau}g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = \frac{1}{2}\left(2g_{\mu\nu}\ddot{x}^\mu\dot{x}^\nu + \,(\dot{x}^\rho\partial_\rho g_{\mu\nu})\dot{x}^\mu\dot{x}^\nu\right) \tag{20}$$

From the definition of the Christoffel symbols $\Gamma_{\mu\nu\rho}$ one sees that if one symmetrises the 1st and 2nd index, 4 of the 6 terms cancel while 2 add up, leading to the useful identity

$$\partial_\rho g_{\mu\nu} = \Gamma_{\mu\nu\rho} + \Gamma_{\nu\mu\rho} \quad . \tag{21}$$

Using this identity together with the fact that $x^\mu(\tau)$ is a solution to the geodesic equation, which means that we also have

$$\ddot{x^\mu} = -\Gamma^\mu_{\nu\rho}\dot{x}^\nu\dot{x}^\rho \tag{22}$$

leaves us with

$$\begin{aligned} \frac{d}{d\tau}\mathcal{L} &= \frac{1}{2}\left(-2g_{\mu\nu}\Gamma^\mu_{\lambda\rho}\dot{x}^\lambda\dot{x}^\rho\dot{x}^\nu + \dot{x}^\rho(\Gamma_{\mu\nu\rho}+\Gamma_{\nu\mu\rho})\dot{x}^\mu\dot{x}^\nu\right) && (23)\\ &= \frac{1}{2}\left(-2\Gamma_{\nu\lambda\rho}\dot{x}^\lambda\dot{x}^\rho\dot{x}^\nu + (\Gamma_{\mu\nu\rho}+\Gamma_{\nu\mu\rho})\dot{x}^\mu\dot{x}^\nu\dot{x}^\rho\right) = 0 && (24)\end{aligned}$$

which is obviously zero if we relabel the indices.

(c) The metric on the 2-sphere is

$$ds^2 = R^2\left(d\theta^2 + \sin^2(\theta)d\phi^2\right) \tag{25}$$

so that in the (θ,ϕ) coordinates we have :

$$g_{\mu\nu} = R^2\begin{pmatrix} 1 & 0 \\ 0 & \sin(\theta)^2 \end{pmatrix} \qquad\qquad g^{\mu\nu} = R^{-2}\begin{pmatrix} 1 & 0 \\ 0 & \sin(\theta)^{-2} \end{pmatrix} \tag{26}$$

Because $g_{\mu\nu}$ is diagonal and $g_{\mu\nu} = g_{\mu\nu}(\theta)$, the only non-vanishing contributions to the Christoffel symbols will come from terms involving $\partial_\theta g_{\phi\phi}$. Keeping this in mind, we first compute the Christoffel symbols with upper index θ,

$$\Gamma^\theta_{\nu\lambda} = \frac{1}{2}g^{\theta\theta}\left(\partial_\lambda g_{\theta\nu} + \partial_\nu g_{\theta\lambda} - \partial_\theta g_{\nu\lambda}\right) = -\frac{1}{2}g^{\theta\theta}\partial_\theta g_{\nu\lambda} \tag{27}$$

and we see that the only non-vanishing term with θ on the top is $\Gamma^\theta_{\phi\phi}$:

$$\Gamma^\theta_{\phi\phi} = -\frac{1}{2}g^{\theta\theta}\partial_\theta g_{\phi\phi} = -\sin(\theta)\cos(\theta) \tag{28}$$

Now if we choose ϕ to be on the top, we get

$$\Gamma^\phi_{\nu\lambda} = \frac{1}{2}g^{\phi\phi}\left(\partial_\lambda g_{\phi\nu} + \partial_\nu g_{\phi\lambda} - \partial_\phi g_{\nu\lambda}\right) = \frac{1}{2}g^{\phi\phi}\left(\partial_\lambda g_{\phi\nu} + \partial_\nu g_{\phi\lambda}\right) \tag{29}$$

so that the only non-vanishing terms with ϕ on the top are $\Gamma^\phi_{\phi\theta} = \Gamma^\phi_{\theta\phi}$:

$$\Gamma^\phi_{\phi\theta} = \frac{1}{2}g^{\phi\phi}\partial_\theta g_{\phi\phi} = \frac{\cos(\theta)}{\sin(\theta)} \tag{30}$$

With these Christoffel symbols we can now write down the geodesic equation $\ddot{x}^\mu + \Gamma^\mu_{\nu\rho}\dot{x}^\nu \dot{x}^\rho$ in the (θ, ϕ) coordinate. We find

$$\begin{cases} \ddot{\theta} + \Gamma^\theta_{\phi\phi}\dot{\phi}\dot{\phi} = \ddot{\theta} - \sin(\theta)\cos(\theta)\dot{\phi}\dot{\phi} = 0 & \mu = \theta \\ \\ \ddot{\phi} + 2\Gamma^\phi_{\theta\phi}\dot{\theta}\dot{\phi} = \ddot{\phi} + 2\dfrac{\cos(\theta)}{\sin(\theta)}\dot{\theta}\dot{\phi} = 0 & \mu = \phi \end{cases} \tag{31}$$

Using the Euler-Lagrange equations with $\mathcal{L} = \frac{1}{2}(\dot{\theta}^2 + \sin(\theta)^2\dot{\phi}^2)$, we get :

$$\begin{cases} \dfrac{d}{d\tau}\dfrac{\partial \mathcal{L}}{\partial \dot{\theta}} - \dfrac{\partial \mathcal{L}}{\partial \theta} = \dfrac{1}{2}\left(\dfrac{d}{d\tau}2\dot{\theta} - 2\sin(\theta)\cos(\theta)\dot{\phi}^2\right) = 0 \\ \\ \dfrac{d}{d\tau}\dfrac{\partial \mathcal{L}}{\partial \dot{\phi}} - \dfrac{\partial \mathcal{L}}{\partial \phi} = \dfrac{1}{2}\dfrac{d}{d\tau}\left(2\sin(\theta)^2\dot{\phi}\right) = \sin(\theta)^2\ddot{\phi} + 2\cos(\theta)\sin(\theta)\dot{\theta}\dot{\phi} = 0 \end{cases} \tag{32}$$

which are the same equations as in (31).

As discussed during the course, conversely one can simply use the Euler-Lagrange equations to read off all the non-zero components of the Christoffel symbol.

We can easily see that the great circles $(\theta(\tau), \phi(\tau)) = (\tau, \phi_0)$ on S^2 are solutions to the equations just found. It is indeed the case because for that particular solution ϕ is constant along a great circle, which implies $\ddot{\phi} = \dot{\phi} = 0$ and simultaneously $\theta(\tau) = \tau$ so that $\ddot{\theta}$ vanishes. By looking at equation (19) or (20) we can see that every curve with $\ddot{\theta} = \ddot{\phi} = \dot{\phi} = 0$ trivially satisfy both equations and therefore such curves are geodesics.