1. Free Relativistic Particle in Arbitrary Coordinates

- From

\[ g_{\mu\nu} = \eta_{ab} J_a^\mu J_b^\nu \]  

one deduces

\[ g_{\mu\nu,\lambda} = \eta_{ab} (J_a^\mu J_b^\nu + J_a^\nu J_b^\mu - \epsilon_{\mu\nu\lambda} J_a^\lambda) \]  

where

\[ J_a^\mu = \partial_\lambda J_a^{\mu\lambda} = \frac{\partial^2 \xi_a}{\partial x^\mu \partial x^\lambda} = J_a^{\mu\lambda}. \]  

- Therefore one has

\[ \Gamma_{\mu\nu\lambda} = \frac{1}{2} (g_{\mu\nu,\lambda} + g_{\mu\lambda,\nu} - g_{\nu\lambda,\mu}) \]

\[ = \frac{1}{2} \eta_{ab} (J_a^\mu J_b^\nu + J_a^\nu J_b^\mu + J_a^\mu J_b^\nu - J_a^\nu J_b^\mu + J_a^\nu J_b^\mu - J_a^\mu J_b^\mu) \]

\[ = \eta_{ab} J_a^\mu J_b^\nu, \]

where the cancellations in passing to the last line arise from the symmetries

\[ \eta_{ab} = \eta_{ba}, J_b^{\mu\nu} = J_b^{\nu\mu} \text{ etc.} \]

- Thus (writing out everything in detail),

\[ \Gamma_{\mu\nu\lambda} = g_{\mu\rho} \Gamma_{\rho\nu\lambda} = \eta^{cd} J_c^\rho J_d^\nu \eta_{ab} J_a^\mu J_b^\lambda = \eta^{cd} J_c^\mu \eta_{ab} J_a^\nu J_b^\lambda \]

\[ = \eta^{ca} J_c^\mu \eta_{ab} J_a^\nu J_b^\lambda = \delta_c^a J_c^\mu = J_b^b J_b^\nu \]

as was to be shown.

2. Geodesics

(a) With the Lagrangian

\[ L(x^\mu, \dot{x}^\mu) = \frac{1}{2} g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = \frac{1}{2} \dot{x}^\mu \dot{x}^\nu \]

where \( g_{\mu\nu} = g_{\mu\nu}(x^\rho) \) one computes

\[ \frac{\partial L}{\partial x^\mu} = \frac{1}{2} \dot{x}^\mu \dot{x}^\nu \partial_\mu g_{\rho\nu} \]  

\[ \frac{\partial L}{\partial \dot{x}^\mu} = g_{\mu\rho} \dot{x}^\rho \partial_\nu g_{\rho\mu} = g_{\mu\rho} \dot{x}^\rho \delta_\nu^\mu = g_{\mu\rho} \dot{x}^\rho \]  

\[ \frac{d}{d\tau} \left( \frac{\partial L}{\partial \dot{x}^\mu} \right) = g_{\mu\rho} \ddot{x}^\rho + \dot{x}^\rho \dot{x}^\nu \partial_\nu g_{\rho\mu} = g_{\mu\rho} \ddot{x}^\rho + \frac{1}{2} (\dot{x}^\rho \dot{x}^\nu \partial_\nu g_{\rho\mu} + \dot{x}^\nu \dot{x}^\rho \partial_\mu g_{\rho\nu}) \]  

Thus the Euler-Lagrange equations become

\[ \left[ \text{E.-L.} \right] = g_{\mu\rho} \ddot{x}^\rho + \frac{1}{2} (\partial_\nu g_{\mu\rho} + \partial_\rho g_{\nu\mu} - \partial_\mu g_{\rho\nu}) \dot{x}^\nu \dot{x}^\rho \]

\[ = g_{\mu\rho} \ddot{x}^\rho + \Gamma_{\mu\nu\rho} \dot{x}^\rho \dot{x}^\nu = 0 \]
and they can be written in the usual (geodesic equation) form by multiplying by $g^{\lambda \mu}$ to move the index $\mu$ up:

$$g^{\lambda \mu} (g_{\mu \nu} \ddot{x}^\mu + \Gamma_{\mu \nu \rho} \dot{x}^\rho \dot{x}^\nu) = \ddot{x}^\lambda + \Gamma_{\rho \nu \lambda} \dot{x}^\rho \dot{x}^\nu = 0 \tag{12}$$

(b) First we compute :

$$\frac{d}{d\tau} \mathcal{L} = \frac{1}{2} \frac{d}{d\tau} g_{\mu \nu} \dot{x}^\mu \dot{x}^\nu = \frac{1}{2} \left( 2 g_{\mu \nu} \ddot{x}^\mu \dot{x}^\nu + (\dot{x}^\rho \partial_\rho g_{\mu \nu}) \dot{x}^\mu \dot{x}^\nu \right) \tag{13}$$

From the definition of the Christoffel symbols $\Gamma_{\mu \nu \rho}$ one sees that if one symmetrises the 1st and 2nd index, 4 of the 6 terms cancel while 2 add up, leading to the useful identity

$$\partial_\rho g_{\mu \nu} = \Gamma_{\mu \nu \rho} + \Gamma_{\nu \mu \rho}. \tag{14}$$

Using this identity together with the fact that $\dot{x}^\mu(\tau)$ is a solution to the geodesic equation, which means that we also have

$$\ddot{x}^\mu = -\Gamma_{\nu \mu \rho} \dot{x}^\nu \dot{x}^\rho \tag{15}$$

leaves us with

$$\frac{d}{d\tau} \mathcal{L} = \frac{1}{2} \left( -2 g_{\mu \nu} \Gamma_{\lambda \mu \nu} \ddot{x}^\lambda \dot{x}^\rho \dot{x}^\nu + \dot{x}^\rho (\Gamma_{\mu \nu \rho} + \Gamma_{\nu \mu \rho}) \ddot{x}^\mu \dot{x}^\nu \right) \tag{16}$$

$$= \frac{1}{2} \left( -2 \Gamma_{\nu \lambda \rho} \ddot{x}^\lambda \dot{x}^\rho \dot{x}^\nu + (\Gamma_{\mu \nu \rho} + \Gamma_{\nu \mu \rho}) \ddot{x}^\mu \dot{x}^\nu \dot{x}^\rho \right) = 0 \tag{17}$$

which is obviously zero if we relabel the indices.

(c) The metric on the 2-sphere is

$$ds^2 = R^2 (d\theta^2 + \sin^2(\theta)d\phi^2) \tag{18}$$

so that in the $(\theta, \phi)$ coordinates we have :

$$g_{\mu \nu} = R^2 \begin{pmatrix} 1 & 0 \\ 0 & \sin(\theta)^2 \end{pmatrix} \quad g^{\mu \nu} = R^{-2} \begin{pmatrix} 1 & 0 \\ 0 & \sin(\theta)^{-2} \end{pmatrix} \tag{19}$$

Because $g_{\mu \nu}$ is diagonal and $g_{\mu \nu} = g_{\mu \nu}(\theta)$, the only non-vanishing contributions to the Christoffel symbols will come from terms involving $\partial_\theta g_{\phi \phi}$. Keeping this in mind, we first compute the Christoffel symbols with upper index $\theta$,

$$\Gamma_{\theta \nu \lambda} = \frac{1}{2} g^{\theta \theta} (\partial_\lambda g_{\theta \nu} + \partial_\nu g_{\theta \lambda} - \partial_\theta g_{\nu \lambda}) = -\frac{1}{2} g^{\theta \theta} \partial_\theta g_{\nu \lambda} \tag{20}$$

and we see that the only non-vanishing term with $\theta$ on the top is $\Gamma_{\phi \phi}^\theta$ :

$$\Gamma_{\phi \phi}^\theta = -\frac{1}{2} g^{\theta \theta} \partial_\theta g_{\phi \phi} = -\sin(\theta) \cos(\theta) \tag{21}$$

Now if we choose $\phi$ to be on the top, we get

$$\Gamma_{\nu \lambda}^\phi = \frac{1}{2} g_{\phi \phi} (\partial_\lambda g_{\phi \nu} + \partial_\nu g_{\phi \lambda} - \partial_\phi g_{\nu \lambda}) = \frac{1}{2} g_{\phi \phi} (\partial_\lambda g_{\phi \nu} + \partial_\nu g_{\phi \lambda}) \tag{22}$$
so that the only non-vanishing terms with $\phi$ on the top are $\Gamma^\phi_{\phi\theta} = \Gamma^\phi_{\theta\phi}$:

$$
\Gamma^\phi_{\phi\theta} = \frac{1}{2} g^{\phi\phi} g_{\theta\phi} = \frac{\cos(\theta)}{\sin(\theta)} \tag{23}
$$

With these Christoffel symbols we can now write down the geodesic equation $\ddot{x}^\mu + \Gamma^\mu_{\nu\rho} \dot{x}^\nu \dot{x}^\rho$ in the $(\theta, \phi)$ coordinate. We find

\[
\begin{cases}
\ddot{\theta} + \Gamma^\theta_{\phi\phi} \dot{\phi} \dot{\phi} = \ddot{\theta} - \sin(\theta) \cos(\theta) \dot{\phi} \dot{\phi} = 0 & \mu = \theta \\
\ddot{\phi} + 2 \Gamma^\phi_{\theta\phi} \dot{\theta} \dot{\phi} = \ddot{\phi} + 2 \frac{\cos(\theta)}{\sin(\theta)} \dot{\theta} \dot{\phi} = 0 & \mu = \phi
\end{cases}
\tag{24}
\]

Using the Euler-Lagrange equations with $L = \frac{1}{2} (\dot{\theta}^2 + \sin(\theta)^2 \dot{\phi}^2)$, we get:

\[
\begin{cases}
\frac{d}{d\tau} \frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = \frac{1}{2} \left( \frac{d}{d\tau} 2 \ddot{\theta} - 2 \sin(\theta) \cos(\theta) \dot{\phi}^2 \right) = 0 \\
\frac{d}{d\tau} \frac{\partial L}{\partial \dot{\phi}} - \frac{\partial L}{\partial \phi} = \frac{1}{2} \frac{d}{d\tau} \left( 2 \sin(\theta)^2 \dot{\phi}^2 \right) = \sin(\theta)^2 \dot{\phi}^2 + 2 \cos(\theta) \sin(\theta) \dot{\theta} \dot{\phi} = 0
\end{cases}
\tag{25}
\]

which are the same equations as in (31).

As discussed during the course, conversely one can simply use the Euler-Lagrange equations to read off all the non-zero components of the Christoffel symbol.

We can easily see that the great circles $(\theta(\tau), \phi(\tau)) = (\tau, \phi_0)$ on $S^2$ are solutions to the equations just found. It is indeed the case because for that particular solution $\phi$ is constant along a great circle, which implies $\ddot{\phi} = \dot{\phi} = 0$ and simultaneously $\theta(\tau) = \tau$ so that $\ddot{\theta}$ vanishes. By looking at equation (19) or (20) we can see that every curve with $\ddot{\theta} = \ddot{\phi} = \dot{\phi} = 0$ trivially satisfy both equations and therefore such curves are geodesics.