Solutions to Assignments 06

1. Properties of the Riemann Curvature Tensor

   (a) Contracting the cyclic symmetry property (first Bianchi identity) of the Riemann tensor over any pair of indices, e.g. $(\alpha\gamma)$, and using the anti-symmetry in the first and second pair of indices, one finds

$$0 = g^{\alpha\gamma}(R_{\alpha\beta\gamma\delta} + R_{\alpha\gamma\delta\beta} + R_{\alpha\delta\beta\gamma}) = R_{\beta\delta} + 0 - R_{\delta\beta} \quad . \tag{1}$$

   (b) Writing $\circlearrowleft$ for the cyclic permutations in (α, β, γ) and then using the cyclic symmetry $R^{\rho}{}_{\alpha\beta\gamma} + \circlearrowleft = 0$, we have

$$\begin{aligned}
[\nabla_\alpha, [\nabla_\beta, \nabla_\gamma]]V^\lambda + \circlearrowleft \quad &= \nabla_\alpha(R^{\lambda}{}_{\rho\beta\gamma}V^\rho) - R^{\lambda}{}_{\rho\beta\gamma}\nabla_\alpha V^\rho + R^{\rho}{}_{\alpha\beta\gamma}\nabla_\rho V^\lambda + \circlearrowleft \\
&= \nabla_\alpha(R^{\lambda}{}_{\rho\beta\gamma})V^\rho + R^{\rho}{}_{\alpha\beta\gamma}\nabla_\rho V^\lambda + \circlearrowleft \\
&= \nabla_\alpha(R^{\lambda}{}_{\rho\beta\gamma})V^\rho + \circlearrowleft \\
&= g^{\lambda\mu}\left[\nabla_\alpha R_{\mu\nu\beta\gamma} + \circlearrowleft\right] V^\nu = 0
\end{aligned} \tag{2}$$

   which gives the desired result.

   (c) Contracting the Bianchi identity over the indices (μ, β) and (ν, α) one finds :

$$\begin{aligned}
g^{\nu\alpha}g^{\mu\beta}\left[\nabla_\alpha R_{\mu\nu\beta\gamma} + \circlearrowleft\right] \quad &= g^{\nu\alpha}g^{\mu\beta}\left[\nabla_\alpha R_{\mu\nu\beta\gamma} + \nabla_\beta R_{\mu\nu\gamma\alpha} + \nabla_\gamma R_{\mu\nu\alpha\beta}\right] \\
&= \nabla_\alpha R^{\alpha\beta}{}_{\beta\gamma} + \nabla_\beta R^{\alpha\beta}{}_{\gamma\alpha} + \nabla_\gamma R^{\alpha\beta}{}_{\alpha\beta} \\
&= -\nabla_\alpha R^{\alpha}{}_{\gamma} - \nabla_\beta R^{\beta}{}_{\gamma} + \nabla_\gamma R \\
&= -\nabla_\alpha\left[2R^{\alpha}{}_{\gamma} - \delta^{\alpha}_{\gamma}R\right] = 0
\end{aligned} \tag{3}$$

   And defining the *Einstein tensor* as $G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R$, we see that the contracted Bianchi identity (4) is equivalent to $\nabla^\alpha G_{\alpha\beta} = 0$ because :

$$\nabla^\alpha G_{\alpha\beta} = \nabla^\alpha(R_{\alpha\beta} - \tfrac{1}{2}g_{\alpha\beta}R) = \frac{1}{2}\nabla_\alpha(2R^{\alpha}{}_{\beta} - g^{\alpha}{}_{\beta}R) \tag{4}$$

   so that $\nabla^\alpha G_{\alpha\beta} = 0 \;\Leftrightarrow\; \nabla_\alpha\left[2R^{\alpha}{}_{\gamma} - \delta^{\alpha}_{\gamma}R\right] = 0$ where we have use the fact that $g^{\alpha}{}_{\beta} = g^{\alpha\lambda}g_{\lambda\beta} = \delta^{\alpha}{}_{\beta}$ simply because $g^{\alpha\lambda}$ is the inverse of $g_{\alpha\lambda}$.

2. On the Klein-Gordon Field in a curved space-time

   The action is

$$S[\phi, g_{\alpha\beta}] = \int \sqrt{g}d^4x\; L \equiv -\tfrac{1}{2}\int \sqrt{g}d^4x\left(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + m^2\phi^2\right) \tag{5}$$

   and the energy-momentum tensor is

$$T_{\alpha\beta} = \partial_\alpha\phi\partial_\beta\phi + g_{\alpha\beta}L \tag{6}$$

(a) To show that the energy-momentum tensor is covariantly conserved we will use $\partial_\mu \phi = \nabla_\mu \phi$, the commutativity $\nabla_\mu \nabla_\nu \phi = \nabla_\nu \nabla_\mu \phi$ of the covariant derivative on scalars, and the fact that ϕ is a solution to the Klein-Gordon equation $\nabla^\mu \nabla_\mu \phi = m^2 \phi$, then the result follows:

$$\begin{aligned} \nabla^\mu T_{\mu\nu} &= \nabla^\mu (\partial_\mu \phi \partial_\nu \phi) + \nabla^\mu (g_{\mu\nu} L) \\ &= \nabla^\mu (\partial_\mu \phi \partial_\nu \phi) - \frac{1}{2} \nabla_\nu (\partial_\lambda \phi \partial^\lambda \phi + m^2 \phi^2) \\ &= \partial_\nu \phi \nabla^\mu \partial_\mu \phi + \partial_\mu \phi \nabla^\mu \partial_\nu \phi - \partial_\lambda \phi \nabla_\nu \partial^\lambda \phi - m^2 \phi \nabla_\nu \phi \\ &= \partial_\nu \phi \, m^2 \phi + \partial_\mu \phi \nabla^\mu \nabla_\nu \phi - \partial_\lambda \phi \nabla^\lambda \nabla_\nu \phi - m^2 \phi \nabla_\nu \phi = 0 \ . \end{aligned} \quad (7)$$

(b) The variation of the action with respect to the metric is

$$\delta S = \int d^4x \, (\delta(\sqrt{g}) L + \sqrt{g} \delta L) = -\tfrac{1}{2} \int d^4x \sqrt{g} \, (g_{\mu\nu} L \delta g^{\mu\nu} - 2\delta L) \quad (8)$$

(valid for any Lagrangian L). Using $\delta(g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi) = (\delta g^{\mu\nu}) \partial_\mu \phi \partial_\nu \phi$, one finds

$$\delta S = -\frac{1}{2} \int d^4x \sqrt{g} (g_{\mu\nu} L + \partial_\mu \phi \partial_\nu \phi) \delta g^{\mu\nu} = -\frac{1}{2} \int d^4x \sqrt{g} T_{\mu\nu} \delta g^{\mu\nu} \quad (9)$$

as claimed.

3. On the Maxwell Equations in Curved Space-Time The action is

$$S[A_\alpha, g_{\alpha\beta}] = \int \sqrt{g} d^4x \; L = -\tfrac{1}{4} \int \sqrt{g} d^4x F_{\alpha\beta} F^{\alpha\beta} \quad (10)$$

and the gauge-invariant and generally covariant energy momentum tensor is

$$T_{\alpha\beta} = F_{\alpha\gamma} F_\beta^{\ \gamma} - \tfrac{1}{4} g_{\alpha\beta} F_{\gamma\delta} F^{\gamma\delta} \quad (11)$$

(a) We compute, using $\nabla_\mu F^{\mu\lambda} = 0$,

$$\begin{aligned} \nabla_\mu T^{\mu\nu} &= \nabla_\mu (F^\mu_{\ \lambda} F^{\nu\lambda} - \tfrac{1}{4} g^{\mu\nu} F_{\lambda\rho} F^{\lambda\rho}) = F^\mu_{\ \lambda} \nabla_\mu F^{\nu\lambda} - \tfrac{1}{2} F_{\lambda\rho} \nabla^\nu F^{\lambda\rho} \\ &= F_{\mu\lambda} \left(\nabla^\mu F^{\nu\lambda} - \tfrac{1}{2} \nabla^\nu F^{\mu\lambda} \right) = \tfrac{1}{2} F_{\mu\lambda} \left(\nabla^\mu F^{\nu\lambda} - \nabla^\mu F^{\lambda\nu} - \nabla^\nu F^{\mu\lambda} \right) \end{aligned} \quad (12)$$

Uinsg the anti-symmetry of F, one can write this as

$$\nabla_\mu T^{\mu\nu} = -\tfrac{1}{2} F_{\mu\lambda} \left(\nabla^\lambda F^{\nu\mu} + \nabla^\mu F^{\lambda\nu} + \nabla^\nu F^{\mu\lambda} \right) = 0 \quad (13)$$

(b) For the metric variation of the action, we can also use the general formula (8). For the variation of the Lagrangian with respect to the metric, we note that

$$\delta(g^{\mu\lambda} g^{\nu\rho} F_{\mu\nu} F_{\lambda\rho}) = 2(\delta g^{\mu\lambda}) g^{\nu\rho} F_{\mu\nu} F_{\lambda\rho} = 2(\delta g^{\mu\nu}) g^{\lambda\rho} F_{\mu\lambda} F_{\nu\rho} = 2(\delta g^{\mu\nu}) F_{\mu\lambda} F_\nu^{\ \lambda} \quad (14)$$

and therefore $-2\delta L = (\delta g^{\mu\nu}) F_{\mu\lambda} F_\nu^{\ \lambda}$. Putting the pieces together one gets (11).