Solutions to Assignments 03

1. Tensor analysis I: Tensor Algebra

   The invariance of $V(x)$ under coordinate transformations follows from the fact that partial derivatives are covectors and that they are contracted with a vector to form the field $V(x)$,

$$V^\alpha \partial_\alpha = J^\alpha_\mu J^\nu_\alpha V^\mu \partial_\nu = \delta^\nu_\mu V^\mu \partial_\nu = V^\mu \partial_\mu \ \ . \tag{1}$$

   Likewise for a covector:

$$dy^\alpha = J^\alpha_\nu dx^\nu \quad \Rightarrow \quad A_\alpha dy^\alpha = J^\mu_\alpha J^\alpha_\nu A_\mu dx^\nu = A_\mu dx^\mu \ \ . \tag{2}$$

2. Tensor Analysis II: the Covariant Derivative

   Consider a covector $A_\mu(x)$ and a coordinate transformation $x^\mu = x^\mu(y^\alpha)$, with Jacobi matrix

$$J^\mu_\alpha = \frac{\partial x^\mu}{\partial y^\alpha} \ \ . \tag{3}$$

   As a covector, A_μ transforms as $A_\alpha = J^\mu_\alpha A_\mu$, and therefore its derivative transforms as (using $\partial_\beta = J^\nu_\beta \partial_\nu$)

$$A_\alpha = J^\mu_\alpha A_\mu \quad \Rightarrow \quad \partial_\beta A_\alpha = J^\mu_\alpha J^\nu_\beta \partial_\nu A_\mu + (\partial_\beta J^\mu_\alpha) A_\mu \ \ . \tag{4}$$

   Because of

$$\partial_\beta J^\mu_\alpha = \frac{\partial^2 x^\mu}{\partial y^\alpha \partial y^\beta} = \partial_\alpha J^\mu_\beta \ \ , \tag{5}$$

   for the anti-symmetrised derivative one finds the tensorial transformation behaviour

$$\partial_\beta A_\alpha - \partial_\alpha A_\beta = J^\mu_\alpha J^\nu_\beta (\partial_\nu A_\mu - \partial_\mu A_\nu) \ \ . \tag{6}$$

   Because the Christoffel symbols are symmetric in their lower indices, they always drop out of the anti-symmetrised derivatives of anti-symmetric covariant tensors. In the present (simplest) case of covectors, one has

$$\nabla_\mu A_\nu - \nabla_\nu A_\mu = \partial_\mu A_\nu - \Gamma^\lambda_{\mu\nu} A_\lambda - \partial_\nu A_\mu + \Gamma^\lambda_{\nu\mu} A_\lambda = \partial_\mu A_\nu - \partial_\nu A_\mu \ \ . \tag{7}$$

3. The Effective Geodesic Potential

   Starting with the metric

$$ds^2 = -f(r)dt^2 + f(r)^{-1} dr^2 + r^2 d\Omega^2 \quad , \quad f(r) = 1 + 2\phi(r) \tag{8}$$

   one implements the following steps:

- the Lagrangian $\mathcal{L}$ is conserved,

$$-f(r)\dot{t}^2 + f(r)^{-1}\dot{r}^2 + r^2(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2) = \epsilon \tag{9}$$

  where $\epsilon = -1, 0$ for massive (massless) particles.

- by spherical symmetry, angular momentum is conserved, thus the motion is planar, and one can choosse the coordinates such that this motion takes place in the equatorial plane $\theta = \pi/2$, $\dot{\theta} = 0$, leading to

$$-f(r)\dot{t}^2 + f(r)^{-1}\dot{r}^2 + r^2\dot{\phi}^2 = \epsilon \quad . \tag{10}$$

- rotational and time-translational symmetry lead to the conserved quantities

$$E = f(r)\dot{t} \qquad L = r^2\dot{\phi} \tag{11}$$

  (energy and angular momentum), and using these equations to eliminate $\dot{t}$ and $\dot{\phi}$ from the Lagrangian, one finds

$$-E^2 f(r)^{-1} + f(r)^{-1}\dot{r}^2 + L^2/r^2 = \epsilon \quad . \tag{12}$$

  Multiplying by $f(r)$ and rearranging, this gives

$$\dot{r}^2 + f(r)L^2/r^2 - \epsilon f(r) = E^2 \quad . \tag{13}$$

- This already has the desired form of an effective Newtonian potential equation, but it is typically more useful to separate the constant (asymptotically Minkowski) part of $f(r)$ from the rest. Thus, with $f(r) = 1 + 2\phi(r)$ one has

$$\tfrac{1}{2}\dot{r}^2 + V_{eff}(r) = E_{eff} \tag{14}$$

  where

$$V_{eff}(r) \equiv V(r) + L^2/2r^2 = \phi(r)(-\epsilon + L^2/r^2) + L^2/2r^2 \tag{15}$$

  and

$$E_{eff} = (E^2 + \epsilon)/2 \tag{16}$$