# Solutions to Assignments 04

1. Tensor Analysis III: The Covariant Divergence and the Laplacian

(a) First we compute $\Gamma^{\mu}_{\mu\lambda}$ with the definition

$$\Gamma^{\mu}_{\mu\lambda} = \frac{1}{2} g^{\mu\rho}(\partial_\mu g_{\rho\lambda} + \partial_\lambda g_{\mu\rho} - \partial_\rho g_{\mu\lambda}) = \frac{1}{2} g^{\mu\rho} \partial_\lambda g_{\mu\rho} \tag{1}$$

(the 1st and 3rd term cancel). Then, we use the relation $g^{-1}\partial_\lambda g = g^{\mu\nu}\partial_\lambda g_{\mu\nu}$ to find that :

$$\Gamma^{\mu}_{\mu\lambda} = \frac{1}{2} g^{\mu\rho}\partial_\lambda g_{\mu\rho} = \frac{1}{2} g^{-1}\partial_\lambda g = g^{-1/2}\partial_\lambda g^{+1/2} \tag{2}$$

where in the last equality we used the fact that $\partial_\lambda g^{+1/2} = \frac{1}{2} g^{-1/2}\partial_\lambda g$.

(b) We can now compute the covariant divergence

$$\nabla_\mu J^\mu = \partial_\mu J^\mu + \Gamma^{\mu}_{\mu\rho} J^\rho = \partial_\mu J^\mu + J^\rho g^{-1/2}\partial_\rho g^{+1/2} = g^{-1/2}\partial_\mu (g^{1/2} J^\mu) \tag{3}$$

and

$$\begin{aligned} \nabla_\mu F^{\mu\nu} &= \partial_\mu F^{\mu\nu} + \Gamma^{\mu}_{\mu\rho} F^{\rho\nu} + \Gamma^{\nu}_{\mu\rho} F^{\mu\rho} = \partial_\mu F^{\mu\nu} + \Gamma^{\mu}_{\mu\rho} F^{\rho\nu} \\ &= \partial_\mu F^{\mu\nu} + F^{\rho\nu} g^{-1/2}\partial_\rho g^{+1/2} = g^{-1/2}\partial_\mu (g^{1/2} F^{\mu\nu}) \end{aligned} \tag{4}$$

where the last term in the first equation vanishes because an antisymmetric tensor ($F^{\mu\rho}$) is contracted with a symmetric object ($\Gamma^{\nu}_{\mu\rho}$).

(c) To calculate the Laplacian, we just need the metric,

$$ds^2 = dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \quad \Leftrightarrow \quad (g_{\alpha\beta}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2\theta \end{pmatrix} \tag{5}$$

its inverse,

$$(g^{\alpha\beta}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^{-2} & 0 \\ 0 & 0 & r^{-2}(\sin\theta)^{-2} \end{pmatrix} \tag{6}$$

and its determinant,

$$g = r^4 \sin^2\theta \quad \Rightarrow \quad \sqrt{g} = r^2 \sin\theta \tag{7}$$

Then one calculates

$$\begin{aligned} \Box\Phi &= \frac{1}{r^2 \sin\theta}\partial_\alpha (r^2 \sin\theta g^{\alpha\beta}\partial_\beta \Phi) \\ &= \frac{1}{r^2\sin\theta}\left(\partial_r(r^2\sin\theta\partial_r\Phi) + \partial_\theta(\sin\theta\partial_\theta\Phi) + \partial_\phi((\sin\theta)^{-1}\partial_\theta\Phi)\right) \\ &= r^{-2}\partial_r(r^2\partial_r\Phi) + r^{-2}\left((\sin\theta)^{-1}\partial_\theta(\sin\theta\partial_\theta\Phi) + (\sin\theta)^{-2}\partial_\phi^2\Phi\right) \end{aligned} \tag{8}$$

This can now be rewritten in many ways, e.g. as

$$\Box = \partial_r^2 + \frac{2}{r}\partial_r + \frac{\Delta_{S^2}}{r^2} \tag{9}$$

with

$$\Delta_{S^2} = \frac{1}{\sin\theta}\partial_a(\sin\theta g^{ab}\partial_b) \tag{10}$$

($x^a = (\theta,\phi)$) the Laplace operator on the unit 2-sphere.

**Remark:** This calculation evidently generalises to any dimension. Thus in spherical coordinates in which the Euclidean metric has the form

$$ds^2 = dr^2 + r^2 d\Omega_n^2 \tag{11}$$

(with $d\Omega_n^2$ the line element on the unit $n$-sphere), the Laplace operator on $\mathbb{R}^{n+1}$ takes the form

$$\Box = \partial_r^2 + \frac{n}{r}\partial_r + \frac{\Delta_{S^n}}{r^2} \tag{12}$$

2. Freely Falling Schwarzschild Observers

For zero angular momentum, and with $\dot{r}_{r=R} = 0$ the effective potential equation reduces to

$$E^2 - 1 = \dot{r}^2 - \frac{2m}{r} \quad \Rightarrow \quad \dot{r}^2 = \frac{2m}{r} - \frac{2m}{R} \ , \tag{13}$$

which integrates to

$$\tau_{R\to r_1} = -(2m)^{-1/2}\int_R^{r_1} dr \ \left(\frac{Rr}{R-r}\right)^{1/2} \ . \tag{14}$$

This integral can be calculated in closed form, e.g. via the change of variables

$$\frac{r}{R} = \sin^2\alpha \qquad \alpha_1 \leq \alpha \leq \frac{\pi}{2} \ , \tag{15}$$

leading to

$$\tau_{R\to r_1} = 2\left(\frac{R^3}{2m}\right)^{1/2}\int_{\alpha_1}^{\pi/2} d\alpha \ \sin^2\alpha = \left(\frac{R^3}{2m}\right)^{1/2}\left[\alpha - \tfrac{1}{2}\sin 2\alpha\right]_{\alpha_1}^{\pi/2} \ . \tag{16}$$

For $r_1 \to 0 \Leftrightarrow \alpha_1 \to 0$ one obtains

$$\tau_{R\to 0} = \left(\frac{R^3}{2m}\right)^{1/2}(\pi/2) = \pi\left(\frac{R^3}{8m}\right)^{1/2} \tag{17}$$

$R$ and $r_S = 2m$ have dimensions of length, thus the quantity above also has dimensions of length, so what we have actually calculated is $c\tau$, not $\tau$. To obtain proper time, we thus need to divide by $c$. Using the approximate values

$$(R)_{\text{sun}} \approx 7\times 10^{10}\text{cm} \quad (2m)_{\text{sun}} \approx 3\times 10^5\text{cm} \quad c \approx 3\times 10^{10}\text{cm s}^{-1} \tag{18}$$

one finds $\tau_{R\to 0} \approx 2\times 10^3$s, which is roughly 30 minutes.