Solutions to Assignments 01

1. Metrics, Line Elements and Coordinate Transformations

   (a) The determinant of the metric is $1 - a^2$,

   $$\det \begin{pmatrix} 1 & a \\ a & 1 \end{pmatrix} = 1 - a^2 \quad . \tag{1}$$

   The metric is non-degenerate unless $a = \pm 1$. In this case, the line element can be written as $ds^2 = d(x \pm y)^2$, which is a 1-dimensional metric for the single coordinate $x \pm y$.

   The linear coordinate transformation implies the same transformation for the differentials,

   $$dx^1 = \sqrt{1-a^2}dy^1 \quad , \quad dx^2 = ady^1 + dy^2 \tag{2}$$

   which in turn implies

   $$\begin{aligned}(dx^1)^2 + (dx^2)^2 &= (1-a^2)(dy^1)^2 + a^2(dy^1)^2 + (dy^2)^2 + 2ady^1dy^2 \\ &= (dy^1)^2 + (dy^2)^2 + 2ady^1dy^2 \quad ,\end{aligned} \tag{3}$$

   as claimed.

   For $a^2 > 1$, set

   $$dt = \sqrt{a^2-1}dy^1 \quad , \quad dx = ady^1 + dy^2 \quad . \tag{4}$$

   Then

   $$-dt^2 = (1-a^2)(dy^1)^2 \tag{5}$$

   and the calculation is now identical to the above.

   (b) There are (at least) 2 ways to do this calculation. The longer one is to use the formula $g_{\mu\nu} = J^a_\mu J^b_\nu \eta_{ab}$ to compute the components of the metric in the new coordinates one by one:

   $$\begin{aligned} g_{TT} &= \eta_{ab}\frac{\partial \xi^a}{\partial T}\frac{\partial \xi^b}{\partial T} = -\left(\frac{\partial t}{\partial T}\right)^2 + \left(\frac{\partial x}{\partial T}\right)^2 \\ &= -X^2\cosh(T)^2 + X^2\sinh(T)^2 = -X^2 \end{aligned} \tag{6}$$

   $$\begin{aligned} g_{TX} &= \eta_{ab}\frac{\partial \xi^a}{\partial T}\frac{\partial \xi^b}{\partial X} = -\frac{\partial t}{\partial T}\frac{\partial t}{\partial X} + \frac{\partial x}{\partial T}\frac{\partial x}{\partial X} \\ &= -X\cosh(T)\sinh(T) + X\sinh(T)\cosh(T) = 0 \end{aligned} \tag{7}$$

   $$\begin{aligned} g_{XX} &= \eta_{ab}\frac{\partial \xi^a}{\partial X}\frac{\partial \xi^b}{\partial X} = -\left(\frac{\partial t}{\partial X}\right)^2 + \left(\frac{\partial x}{\partial X}\right)^2 \\ &= -\sinh(T)^2 + \cosh(T)^2 = 1 \end{aligned} \tag{8}$$

Thus, assembling the results, we can read off that the Minkowski line-element in Rindler coordinates takes the form

$$ds^2 = -X^2 dT^2 + dX^2 \tag{9}$$

Alternatively (and this is frequently the calculationally more efficient way of proceeding, even though in the present example it makes hardly any difference), one can simply calculate dt and dx in terms of the new coordinates once and for all, and then plug the result into the line element to read off all the components of the metric at once:

$$\begin{aligned} t &= X \sinh T \quad , \quad x = X \cosh T \\ dt &= dX \;\sinh T + X \cosh T \; dT \quad , \quad dx = dX \;\cosh T + X \sinh T \; dT \\ ds^2 &= -(dX \;\sinh T + X \cosh T \; dT)^2 + (dX \;\cosh T + X \sinh T \; dT)^2 \\ &= -X^2 dT^2 + dX^2 \quad . \end{aligned} \tag{10}$$

The lines of constant $T = T_0$ satisfy $t = (\tanh T_0)x$. These are straight lines through the origin. The lines of constant $X = X_0$ satisfy $x^2 - t^2 = (X_0)^2$. These are hyperbolae.

2. Geodesics I

(a) The Lagrangian is

$$\mathcal{L} = \tfrac{1}{2} g_{\alpha\beta} \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau} \equiv g_{\alpha\beta} \dot{x}^\alpha \dot{x}^\beta \quad . \tag{11}$$

Thus the Euler-Lagrange equations

$$\frac{d}{d\tau} \frac{\partial \mathcal{L}}{\partial \dot{x}^\gamma} - \frac{\partial \mathcal{L}}{\partial x^\gamma} = 0 \tag{12}$$

are

$$\frac{d}{d\tau} \left(g_{\gamma\beta} \dot{x}^\beta \right) = \tfrac{1}{2} g_{\alpha\beta,\gamma} \dot{x}^\alpha \dot{x}^\beta \tag{13}$$

Using

$$\frac{d}{d\tau} g_{\gamma\beta} = g_{\gamma\beta,\alpha} \dot{x}^\alpha \quad , \tag{14}$$

the terms involving first derivatives of the metric cooperatively combine into the Christoffel symbols,

$$\left(\frac{d}{d\tau} g_{\gamma\beta} \right) \dot{x}^\beta - \tfrac{1}{2} g_{\alpha\beta,\gamma} \dot{x}^\alpha \dot{x}^\beta = g_{\gamma\beta,\alpha} \dot{x}^\alpha \dot{x}^\beta - \tfrac{1}{2} g_{\alpha\beta,\gamma} \dot{x}^\alpha \dot{x}^\beta = \Gamma_{\gamma\alpha\beta} \dot{x}^\alpha \dot{x}^\beta \quad . \tag{15}$$

Here we have used the fact that we can write

$$g_{\gamma\beta,\alpha} \dot{x}^\alpha \dot{x}^\beta = \tfrac{1}{2} \left(g_{\gamma\beta,\alpha} + g_{\gamma\alpha,\beta} \right) \dot{x}^\alpha \dot{x}^\beta \tag{16}$$

because $\dot{x}^\alpha \dot{x}^\beta = \dot{x}^\beta \dot{x}^\alpha$ is symmetric. Therefore one has

$$g_{\gamma\beta}\ddot{x}^\beta + \Gamma_{\gamma\alpha\beta}\dot{x}^\alpha \dot{x}^\beta = 0 \quad . \tag{17}$$

By raising the index (or multiplying with the inverse metric) one can write this as

$$\ddot{x}^\gamma + \Gamma^\gamma_{\ \alpha\beta}\dot{x}^\alpha \dot{x}^\beta = 0 \quad . \tag{18}$$

(b) Using (again)

$$\frac{d}{d\tau} g_{\alpha\beta} = g_{\alpha\beta,\gamma}\dot{x}^\gamma \tag{19}$$

we have

$$\frac{d}{d\tau}\mathcal{L} = g_{\alpha\beta,\gamma}\dot{x}^\alpha \dot{x}^\beta \dot{x}^\gamma + 2 g_{\alpha\beta}\ddot{x}^\alpha \dot{x}^\beta \tag{20}$$

One has the identity

$$g_{\alpha\beta,\gamma} = \Gamma_{\alpha\beta\gamma} + \Gamma_{\beta\alpha\gamma} \tag{21}$$

which implies

$$g_{\alpha\beta,\gamma}\dot{x}^\alpha \dot{x}^\beta \dot{x}^\gamma = 2\Gamma_{\beta\alpha\gamma}\dot{x}^\alpha \dot{x}^\beta \dot{x}^\gamma \tag{22}$$

Thus

$$\frac{d}{d\tau}\mathcal{L} = 2\left(g_{\beta\alpha}\ddot{x}^\alpha + \Gamma_{\beta\alpha\gamma}\dot{x}^\alpha \dot{x}^\gamma\right)\dot{x}^\beta \tag{23}$$

which vanishes precisely for a solution ot the geodesic equations.

The underlying symmetry responsible for the existence of this conserved quantity is of course τ-translation invariance (as the Lagrangian does not depend explicitly on the parameter τ, the corresponding "energy" is conserved).