Solutions to Assignments 04

1. Tensor Analysis II: The Covariant Divergence and the Laplacian

   (a) The covariant divergence is $\nabla_\mu V^\mu = \partial_\mu V^\mu + \Gamma^\mu_{\mu\lambda} V^\lambda$ where

   $$\Gamma^\mu_{\mu\lambda} = \frac{1}{2} g^{\mu\rho}(\partial_\mu g_{\rho\lambda} + \partial_\lambda g_{\mu\rho} - \partial_\rho g_{\mu\lambda}) = \frac{1}{2} g^{\mu\rho}\partial_\lambda g_{\mu\rho} \tag{1}$$

   (the 1st and 3rd term cancel). Now we use $g^{-1}\partial_\lambda g = g^{\mu\nu}\partial_\lambda g_{\mu\nu}$ to find

   $$\Gamma^\mu_{\mu\lambda} = \frac{1}{2} g^{\mu\rho}\partial_\lambda g_{\mu\rho} = \frac{1}{2} g^{-1}\partial_\lambda g = g^{-1/2}\partial_\lambda g^{+1/2} \tag{2}$$

   where in the last equality we used the fact that $\partial_\lambda g^{+1/2} = \frac{1}{2} g^{-1/2}\partial_\lambda g$. We can now compute the covariant divergence

   $$\nabla_\mu J^\mu = \partial_\mu J^\mu + \Gamma^\mu_{\mu\lambda} J^\lambda = \partial_\mu J^\mu + J^\lambda g^{-1/2}\partial_\lambda g^{+1/2} = g^{-1/2}\partial_\mu (g^{1/2} J^\mu) \ . \tag{3}$$

   (b) Analogously, for the covariant divergence of an anti-symmetric $(2,0)$-tensor $F^{\mu\nu}$ one has, using (3),

   $$\begin{aligned} \nabla_\mu F^{\mu\nu} &= \partial_\mu F^{\mu\nu} + \Gamma^\mu_{\mu\rho} F^{\rho\nu} + \Gamma^\nu_{\mu\rho} F^{\mu\rho} = \partial_\mu F^{\mu\nu} + \Gamma^\mu_{\mu\rho} F^{\rho\nu} \\ &= \partial_\mu F^{\mu\nu} + F^{\rho\nu} g^{-1/2}\partial_\rho g^{+1/2} = g^{-1/2}\partial_\mu (g^{1/2} F^{\mu\nu}) \end{aligned} \tag{4}$$

   where the last term in the first equation vanishes because an antisymmetric tensor ($F^{\mu\rho}$) is contracted with a symmetric object ($\Gamma^\nu_{\mu\rho}$).
   Finally, using the result (3) and $\nabla_\beta f = \partial_\beta f$ the Laplacian can be written as

   $$\Box f = \nabla_\alpha (g^{\alpha\beta}\nabla_\beta f) = g^{-1/2}\partial_\alpha (g^{1/2} g^{\alpha\beta}\partial_\beta f) \ . \tag{5}$$

2. Static Schwarzschild Observers in Eddington-Finkelstein Coordinates

   In ingoing EF coordinates, the Schwarschild metric takes the form

   $$ds^2 = -f(r)dv^2 + 2dvdr + r^2 d\Omega^2 \tag{6}$$

   where $v = t + r_*$ satisfies

   $$dv = dt + dr_* = dt + dr/f(r) \quad \Rightarrow \quad \frac{\partial v}{\partial t} = 1 \quad , \quad \frac{\partial v}{\partial r} = f(r)^{-1} \ . \tag{7}$$

   A static observer has 4-velocity

   $$(u^\alpha)_{EF} = (u^v, u^r, u^\theta, u^\phi) = (u^v, 0, 0, 0) \ , \tag{8}$$

   with

   $$g_{\alpha\beta} u^\alpha u^\beta = -f(r)(u^v)^2 = -1 \quad \Rightarrow \quad u^v = f(r)^{-1/2} \ . \tag{9}$$

Thus

$$(u^\alpha)_{EF} = (f(r)^{-1/2}, 0, 0, 0) \quad . \tag{10}$$

This agrees with the result in SS coordinates, as could have also been deduced from the vectorial transformation behaviour (y^μ are now the SS coordinates)

$$(u^v)_{EF} = \frac{\partial v}{\partial y^\mu}(u^\mu)_{SS} = \frac{\partial v}{\partial t}(u^t)_{SS} = (u^t)_{SS} \quad . \tag{11}$$

(a) Since $\dot{u}^\alpha = 0$ for a static oberver, the acceleration is

$$a^\alpha = \Gamma^\alpha_{\beta\gamma} u^\beta u^\gamma = \Gamma^\alpha_{vv}(u^v)^2 = f(r)^{-1}\Gamma^\alpha_{vv} \quad . \tag{12}$$

Even though Christoffel symbols are non-tensorial in general, they do transform in a simple way under the very simple coordinate transformation between SS and EF coordinates. Nevertheless it is more convenient (and in any case a good exercise) to just calculate the relevant Christoffel symbols directly in EF coordinates rather than transforming them from the result in SS coordinates.

First of all, the Christoffel symbol $\Gamma_{\alpha vv}$ is rather obviously non-zero only for $\alpha = r$, and

$$\Gamma_{rvv} = -\tfrac{1}{2} g_{vv,r} = +\tfrac{1}{2} f'(r) = m/r^2 \quad . \tag{13}$$

To determine Γ^α_{vv} we need the components of the inverse metric. Because the metric is not diagonal, this requires a bit of care. In matrix form, the (v, r)-components of the metric and its inverse are

$$(g_{\alpha\beta}) = \begin{pmatrix} -f & 1 \\ 1 & 0 \end{pmatrix} \quad , \quad (g^{\alpha\beta}) = \begin{pmatrix} 0 & 1 \\ 1 & f \end{pmatrix} \tag{14}$$

Therefore, there are two non-vanishing Christoffel symbols Γ^α_{vv}, namely

$$\Gamma^v_{vv} = \Gamma_{rvv} \quad , \quad \Gamma^r_{vv} = f(r)\Gamma_{rvv} \quad , \tag{15}$$

and the acceleration vector is

$$(a^\alpha)_{EF} = (a^v = f(r)^{-1} m/r^2, a^r = m/r^2, 0, 0) \quad . \tag{16}$$

Thus the (non-singular, "Newtonian") r-component agrees with that of the acceleration in SS coordinates, but in addition in EF coordinates there is a v-component which is singular as $r \to 2m$.

(b) Alternatively, and more quickly, since a^α is a vector, one can obtain this result by transforming the result in SS coordinates to EF coordinates,

$$(a^\alpha)_{EF} = \frac{\partial x^\alpha}{\partial y^\mu}(a^\mu)_{SS} = \frac{\partial x^\alpha}{\partial r}(a^r)_{SS} \quad \Rightarrow \quad \begin{cases} (a^v)_{EF} = f(r)^{-1} m/r^2 \\ (a^r)_{EF} = m/r^2 \end{cases} \tag{17}$$

(c) Finally, the norm of the acceleration in EF coordinates is

$$g_{\alpha\beta} a^\alpha a^\beta = -f(r)(a^v)^2 + 2a^v a^r = f(r)^{-1}(m/r^2)^2 \quad , \tag{18}$$

in complete agreement with the result in SS coordinates (as it should).