Solutions to Assignments 05

1. On the Klein-Gordon Field in a curved space-time

   The action is

$$S[\phi, g_{\alpha\beta}] = \int \sqrt{g} d^4x\, L \equiv -\tfrac{1}{2} \int \sqrt{g} d^4x \left(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + m^2\phi^2\right) \tag{1}$$

   and the energy-momentum tensor is

$$T_{\alpha\beta} = \partial_\alpha\phi\partial_\beta\phi + g_{\alpha\beta}L \tag{2}$$

   (a) To show that the energy-momentum tensor is covariantly conserved we will use $\partial_\mu\phi = \nabla_\mu\phi$, the commutativity $\nabla_\mu\nabla_\nu\phi = \nabla_\nu\nabla_\mu\phi$ of the covariant derivative on scalars, and the fact that ϕ is a solution to the Klein-Gordon equation $\nabla^\mu\nabla_\mu\phi = m^2\phi$, then the result follows:

$$\begin{aligned}
\nabla^\mu T_{\mu\nu} &= \nabla^\mu(\partial_\mu\phi\partial_\nu\phi) + \nabla^\mu(g_{\mu\nu}L) \\
&= \nabla^\mu(\partial_\mu\phi\partial_\nu\phi) - \frac{1}{2}\nabla_\nu(\partial_\lambda\phi\partial^\lambda\phi + m^2\phi^2) \\
&= \partial_\nu\phi\nabla^\mu\partial_\mu\phi + \partial_\mu\phi\nabla^\mu\partial_\nu\phi - \partial_\lambda\phi\nabla_\nu\partial^\lambda\phi - m^2\phi\nabla_\nu\phi \\
&= \partial_\nu\phi\, m^2\phi + \partial_\mu\phi\nabla^\mu\nabla_\nu\phi - \partial_\lambda\phi\nabla^\lambda\nabla_\nu\phi - m^2\phi\nabla_\nu\phi = 0\ .
\end{aligned} \tag{3}$$

   (b) The variation of the action with respect to the metric is

$$\delta S = \int d^4x\,(\delta(\sqrt{g})L + \sqrt{g}\delta L) = -\tfrac{1}{2}\int d^4x\sqrt{g}\,(g_{\mu\nu}L\delta g^{\mu\nu} - 2\delta L) \tag{4}$$

   (valid for any Lagrangian L). Using $\delta(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi) = (\delta g^{\mu\nu})\partial_\mu\phi\partial_\nu\phi$, one finds

$$\delta S = -\frac{1}{2}\int d^4x\sqrt{g}(g_{\mu\nu}L + \partial_\mu\phi\partial_\nu\phi)\delta g^{\mu\nu} = -\frac{1}{2}\int d^4x\sqrt{g}T_{\mu\nu}\delta g^{\mu\nu} \tag{5}$$

   as claimed.

2. On the Maxwell Equations in Curved Space-Time The action is

$$S[A_\alpha, g_{\alpha\beta}] = \int \sqrt{g} d^4x\, L = -\tfrac{1}{4}\int \sqrt{g} d^4x F_{\alpha\beta}F^{\alpha\beta} \tag{6}$$

   and the gauge-invariant and generally covariant energy momentum tensor is

$$T_{\alpha\beta} = F_{\alpha\gamma}F_\beta^{\ \gamma} - \tfrac{1}{4}g_{\alpha\beta}F_{\gamma\delta}F^{\gamma\delta} \tag{7}$$

   (a) We compute, using $\nabla_\mu F^{\mu\lambda} = 0$,

$$\begin{aligned}
\nabla_\mu T^{\mu\nu} &= \nabla_\mu(F^\mu_{\ \lambda}F^{\nu\lambda} - \tfrac{1}{4}g^{\mu\nu}F_{\lambda\rho}F^{\lambda\rho}) = F^\mu_{\ \lambda}\nabla_\mu F^{\nu\lambda} - \tfrac{1}{2}F_{\lambda\rho}\nabla^\nu F^{\lambda\rho} \\
&= F_{\mu\lambda}\left(\nabla^\mu F^{\nu\lambda} - \tfrac{1}{2}\nabla^\nu F^{\mu\lambda}\right) = \tfrac{1}{2}F_{\mu\lambda}\left(\nabla^\mu F^{\nu\lambda} - \nabla^\mu F^{\lambda\nu} - \nabla^\nu F^{\mu\lambda}\right)
\end{aligned} \tag{8}$$

   Uinsg the anti-symmetry of F, one can write this as

$$\nabla_\mu T^{\mu\nu} = -\tfrac{1}{2}F_{\mu\lambda}\left(\nabla^\lambda F^{\nu\mu} + \nabla^\mu F^{\lambda\nu} + \nabla^\nu F^{\mu\lambda}\right) = 0 \tag{9}$$

(b) For the metric variation of the action, we can also use the general formula (4). For the variation of the Lagrangian with respect to the metric, we note that

$$\delta(g^{\mu\lambda}g^{\nu\rho}F_{\mu\nu}F_{\lambda\rho}) = 2(\delta g^{\mu\lambda})g^{\nu\rho}F_{\mu\nu}F_{\lambda\rho} = 2(\delta g^{\mu\nu})g^{\lambda\rho}F_{\mu\lambda}F_{\nu\rho} = 2(\delta g^{\mu\nu})F_{\mu\lambda}F_{\nu}{}^{\lambda} \tag{10}$$

and therefore $-2\delta L = (\delta g^{\mu\nu})F_{\mu\lambda}F_{\nu}{}^{\lambda}$. Thus (7) follows.

Painlevé-Gullstrand Coordinates for the Schwarzschild Space-Time

(a) We make a coordinate transformation on the standard Schwarzschild metric with coordinates (t,r) defining a new coordinate $T(t,r) = t + \psi(r)$. This leads us to rewrite the metric with $dT = dt + \psi' dr$ and we find

$$ds^2 = -f(r)dT^2 + 2f(r)\psi' dT dr + f(r)^{-1}(1 - f(r)^2\psi'^2)dr^2 + r^2 d\Omega^2 \tag{11}$$

Choosing $C(r) = f(r)\psi'$ gives the desired result and the function $C(r)$ is completely arbitrary because $\psi(r)$ is arbitrary.

(b) For the Painlevé-Gullstrand coordinate we make a particular choice for $C(r)$ namely $C(r) = \sqrt{1-f(r)}$ such that $g_{rr} = f(r)^{-1}(1 - C(r)^2) = 1$. We are thus left with the metric

$$ds^2 = -f(r)dT^2 + 2\sqrt{\frac{2m}{r}}dT dr + dr^2 + r^2 d\Omega^2 \tag{12}$$

Now, with this new choice of coordinate we sees that any component $g_{\mu\nu}$ of the metric stays finite for any value of $r > 0$ (this was not the case at the beginning in the (t,r)-coordinates). In addition to that we also notice that the determinant of the metric is :

$$\det(g_{\mu\nu}) = (-f(r) - \frac{2m}{r})r^4 \sin(\theta)^2 = -r^4 \sin(\theta)^2 \tag{13}$$

which is non-vanishing for any $r > 0$ (with $\theta \neq 0, \pi$ of course).

(c) If we now make the choice $C(r) = 1$, then the metric becomes :

$$ds^2 = -f(r)dT^2 + 2dTdr + r^2 d\Omega^2 \tag{14}$$

and if we rename $T(t,r)$ to $v(t,r)$, then

$$ds^2 = -f(r)dv^2 + 2dvdr + r^2 d\Omega^2 \tag{15}$$

is exactly the metric in the Eddington-Finkelstein coordinates.

We can also check explicitly that the coordinate transformation is indeed also the same. The particular choice $C(r) = 1$ implies that $\psi(r)$ is such that

$$C(r) = 1 \quad \Leftrightarrow \quad \psi'(r) = \frac{1}{f(r)} \quad \Leftrightarrow \quad \psi(r) = r^* + c \ , \tag{16}$$

where the constant c can be set to zero so that we have

$$T(t,r) = t + \psi(r) = t + r^* = v(t,r) \tag{17}$$

as we should.