Solutions to Assignments 06

1. Riemann Curvature Tensor and Differential Identities

   (a) Writing $\circlearrowleft$ for the cyclic permutations in (α, β, γ) and then using the cyclic symmetry $R^{\rho}{}_{\alpha\beta\gamma} + \circlearrowleft = 0$, we have (for any V)

$$\begin{aligned} 0 &= [\nabla_\alpha, [\nabla_\beta, \nabla_\gamma]]V^\lambda + \circlearrowleft \\ &= \nabla_\alpha (R^\lambda{}_{\rho\beta\gamma} V^\rho) - R^\lambda{}_{\rho\beta\gamma} \nabla_\alpha V^\rho + R^\rho{}_{\alpha\beta\gamma} \nabla_\rho V^\lambda + \circlearrowleft \\ &= (\nabla_\alpha R^\lambda{}_{\rho\beta\gamma}) V^\rho + R^\rho{}_{\alpha\beta\gamma} \nabla_\rho V^\lambda + \circlearrowleft = (\nabla_\alpha R^\lambda{}_{\rho\beta\gamma}) V^\rho + \circlearrowleft \\ \Leftrightarrow \quad & \nabla_\alpha R_{\lambda\rho\beta\gamma} + \circlearrowleft = 0 \ . \end{aligned} \tag{1}$$

   (b) Contracting the Bianchi identity over the indices (μ, β) and (ν, α) one finds

$$\begin{aligned} 0 &= g^{\mu\alpha} g^{\nu\beta} \left[\nabla_\alpha R_{\mu\nu\beta\gamma} + \nabla_\beta R_{\mu\nu\gamma\alpha} + \nabla_\gamma R_{\mu\nu\alpha\beta}\right] \\ &= \nabla_\alpha R^{\alpha\beta}{}_{\beta\gamma} + \nabla_\beta R^{\alpha\beta}{}_{\gamma\alpha} + \nabla_\gamma R^{\alpha\beta}{}_{\alpha\beta} \\ &= -\nabla_\alpha R^\alpha{}_\gamma - \nabla_\beta R^\beta{}_\gamma + \nabla_\gamma R = -\nabla_\alpha \left[2R^\alpha{}_\gamma - \delta^\alpha_\gamma R\right] \\ \Leftrightarrow \quad & \nabla^\alpha G_{\alpha\beta} \equiv \nabla^\alpha (R_{\alpha\beta} - \tfrac{1}{2} g_{\alpha\beta} R) = 0 \end{aligned} \tag{2}$$

   (c) The identity $[\nabla_\alpha, \nabla_\beta] T^{\gamma\delta} = R^\gamma_{\epsilon\alpha\beta} T^{\epsilon\delta} + R^\delta_{\epsilon\alpha\beta} T^{\gamma\epsilon}$ implies

$$[\nabla_\alpha, \nabla_\beta] T^{\alpha\beta} = R^\alpha_{\epsilon\alpha\beta} T^{\epsilon\beta} + R^\beta_{\epsilon\alpha\beta} T^{\alpha\epsilon} = R_{\epsilon\beta} T^{\epsilon\beta} - R_{\epsilon\alpha} T^{\alpha\epsilon} = R_{\alpha\beta}(T^{\alpha\beta} - T^{\beta\alpha}) = 0 \tag{3}$$

   because $R_{\alpha\beta} = R_{\beta\alpha}$ is symmetric.

   (d) Because of this identity and the fact that $F^{\alpha\beta} = -F^{\beta\alpha}$, one has

$$-\nabla_\beta J^\beta = \nabla_\beta \nabla_\alpha F^{\alpha\beta} = \tfrac{1}{2} [\nabla_\beta, \nabla_\alpha] F^{\alpha\beta} = 0 \ . \tag{4}$$

2. Curvature of a class of 2-dimensional Metrics

   The Lagrangian $\mathcal{L} = (\dot{x}^2 + f(x)^2 \dot{\phi}^2)/2$ implies the Euler-Lagrange equations $\ddot{x} - f f' \dot{\phi}^2 = 0$ and $\ddot{\phi} + 2(f'/f)\dot{x}\dot{\phi} = 0$, where $f' \equiv df/dx$. From this one reads off the non-zero Christoffel symbols $\Gamma^x_{\phi\phi} = -ff', \Gamma^\phi_{x\phi} = \Gamma^\phi_{\phi x} = f'/f$, and thus

$$\begin{aligned} R_{x\phi x\phi} &= R^x_{\phi x\phi} = \partial_x \Gamma^x_{\phi\phi} - \partial_\phi \Gamma^x_{\phi x} + \Gamma^x_{x\alpha} \Gamma^\alpha_{\phi\phi} - \Gamma^x_{\phi\alpha} \Gamma^\alpha_{\phi x} \\ &= \partial_x \Gamma^x_{\phi\phi} - \Gamma^x_{\phi\phi} \Gamma^\phi_{\phi x} = -(f')^2 - f f'' + (f f')(f'/f) = -f f'' \\ \Rightarrow \quad & R^\phi_{x\phi x} = g^{\phi\alpha} R_{\alpha x\phi x} = g^{\phi\phi} R_{\phi x\phi x} = R_{\phi x\phi x}/f^2 = R_{x\phi x\phi}/f^2 = -f''/f \\ \Rightarrow \quad & R_{xx} = R^\alpha_{x\alpha x} = R^\phi_{x\phi x} = -f''/f \quad , \quad R_{\phi\phi} = R^x_{\phi x\phi} = -f f'' \quad , \quad R_{x\phi} = 0 \\ \Rightarrow \quad & R_{\alpha\beta} = -(f''/f) g_{\alpha\beta} \quad \Rightarrow \quad R = -2f''/f \end{aligned} \tag{5}$$