Solutions to Assignments 05

1. Tensor Analysis II: The Covariant Divergence and the Laplacian

   (a) The covariant divergence is $\nabla_\mu V^\mu = \partial_\mu V^\mu + \Gamma^\mu_{\mu\lambda} V^\lambda$ where

$$\Gamma^\mu_{\mu\lambda} = \frac{1}{2} g^{\mu\rho}(\partial_\mu g_{\rho\lambda} + \partial_\lambda g_{\mu\rho} - \partial_\rho g_{\mu\lambda}) = \frac{1}{2} g^{\mu\rho} \partial_\lambda g_{\mu\rho} \tag{1}$$

   (the 1st and 3rd term cancel). Now we use $g^{-1}\partial_\lambda g = g^{\mu\nu}\partial_\lambda g_{\mu\nu}$ to find

$$\Gamma^\mu_{\mu\lambda} = \frac{1}{2} g^{\mu\rho}\partial_\lambda g_{\mu\rho} = \frac{1}{2} g^{-1}\partial_\lambda g = g^{-1/2}\partial_\lambda g^{+1/2} \tag{2}$$

   where in the last equality we used the fact that $\partial_\lambda g^{+1/2} = \frac{1}{2} g^{-1/2}\partial_\lambda g$. We can now compute the covariant divergence

$$\nabla_\mu J^\mu = \partial_\mu J^\mu + \Gamma^\mu_{\mu\lambda} J^\lambda = \partial_\mu J^\mu + J^\lambda g^{-1/2}\partial_\lambda g^{+1/2} = g^{-1/2}\partial_\mu (g^{1/2} J^\mu) \ . \tag{3}$$

   (b) Analogously, for the covariant divergence of an anti-symmetric $(2,0)$-tensor $F^{\mu\nu}$ one has, using (3),

$$\begin{aligned} \nabla_\mu F^{\mu\nu} &= \partial_\mu F^{\mu\nu} + \Gamma^\mu_{\mu\rho} F^{\rho\nu} + \Gamma^\nu_{\mu\rho} F^{\mu\rho} = \partial_\mu F^{\mu\nu} + \Gamma^\mu_{\mu\rho} F^{\rho\nu} \\ &= \partial_\mu F^{\mu\nu} + F^{\rho\nu} g^{-1/2}\partial_\rho g^{+1/2} = g^{-1/2}\partial_\mu (g^{1/2} F^{\mu\nu}) \end{aligned} \tag{4}$$

   where the last term in the first equation vanishes because an antisymmetric tensor ($F^{\mu\rho}$) is contracted with a symmetric object ($\Gamma^\nu_{\mu\rho}$).

   Finally, using the result (3) and $\nabla_\beta f = \partial_\beta f$ the Laplacian can be written as

$$\Box f = \nabla_\alpha (g^{\alpha\beta}\nabla_\beta f) = g^{-1/2}\partial_\alpha (g^{1/2} g^{\alpha\beta}\partial_\beta f) \ . \tag{5}$$

   (c) To calculate the Laplacian on the 2-sphere S^2, we just need the metric

$$ds^2 = g_{ab} dx^a dx^b = d\theta^2 + \sin^2\theta d\phi^2 \tag{6}$$

   (with $x^a = (\theta, \phi)$). Then $\sqrt{g} = \sin\theta$, the non-zero components of the inverse metric are $g^{\theta\theta} = 1, g^{\phi\phi} = (\sin\theta)^{-2}$. Therefore (for once writing out everything in perhaps more detail than needed)

$$\begin{aligned} \Delta_{S^2} &= \frac{1}{\sin\theta}\partial_a (\sin\theta g^{ab}\partial_b) \\ &= \frac{1}{\sin\theta}\partial_\theta (\sin\theta g^{\theta\theta}\partial_\theta) + \frac{1}{\sin\theta}\partial_\phi (\sin\theta g^{\phi\phi}\partial_\phi) \\ &= \frac{1}{\sin\theta}\partial_\theta (\sin\theta \partial_\theta) + \frac{1}{\sin\theta}\partial_\phi ((\sin\theta)^{-1}\partial_\phi) \\ &= \partial_\theta^2 + \cot\theta\partial_\theta + (\sin\theta)^{-2}\partial_\phi^2 \ . \end{aligned} \tag{7}$$

The Euclidean metric on $\mathbb{R}^3$ in spherical coordinates $\{x^\alpha\} = \{r, \theta, \phi\} = \{r, x^a\}$ is

$$ds^2 = dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \quad \Leftrightarrow \quad (g_{\alpha\beta}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{pmatrix} \tag{8}$$

with inverse,

$$(g^{\alpha\beta}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^{-2} & 0 \\ 0 & 0 & r^{-2}(\sin\theta)^{-2} \end{pmatrix} \tag{9}$$

(note that the angular components have an additional factor of r^{-2} relative to those of the metric on the 2-sphere) and determinant,

$$g = r^4 \sin^2\theta \quad \Rightarrow \quad \sqrt{g} = r^2 \sin\theta \tag{10}$$

It then follows that the Laplace operator on $\mathbb{R}^3$ in spherical coordinates is

$$\begin{aligned} \Delta &= \frac{1}{r^2\sin\theta}\partial_\alpha(r^2\sin\theta g^{\alpha\beta}\partial_\beta) \\ &= \frac{1}{r^2\sin\theta}\Big(\partial_r(r^2\sin\theta\partial_r) + \partial_a(r^2\sin\theta g^{ab}\partial_b)\Big) \\ &= \partial_r^2 + \frac{2}{r}\partial_r + \frac{1}{r^2}\Delta_{S^2} \quad . \end{aligned} \tag{11}$$

Remark: This calculation evidently generalises to any dimension. Thus in spherical coordinates in which the Euclidean metric has the form

$$ds^2 = dr^2 + r^2 d\Omega_n^2 \tag{12}$$

(with $d\Omega_n^2$ the line element on the unit n-sphere), the Laplace operator on $\mathbb{R}^{n+1}$ takes the form

$$\Box = \partial_r^2 + \frac{n}{r}\partial_r + \frac{\Delta_{S^n}}{r^2} \tag{13}$$

2. Static Schwarzschild Observers in Eddington-Finkelstein Coordinates

In ingoing EF coordinates, the Schwarschild metric takes the form

$$ds^2 = -f(r)dv^2 + 2dvdr + r^2 d\Omega^2 \tag{14}$$

where $v = t + r_*$ satisfies

$$dv = dt + dr_* = dt + dr/f(r) \quad \Rightarrow \quad \frac{\partial v}{\partial t} = 1 \quad , \quad \frac{\partial v}{\partial r} = f(r)^{-1} \quad . \tag{15}$$

A static observer has 4-velocity

$$(u^\alpha)_{EF} = (u^v, u^r, u^\theta, u^\phi) = (u^v, 0, 0, 0) \quad , \tag{16}$$

with

$$g_{\alpha\beta}u^\alpha u^\beta = -f(r)(u^v)^2 = -1 \quad \Rightarrow \quad u^v = f(r)^{-1/2} \ . \tag{17}$$

Thus

$$(u^\alpha)_{EF} = (f(r)^{-1/2}, 0, 0, 0) \ . \tag{18}$$

This agrees with the result in SS coordinates, as could have also been deduced from the vectorial transformation behaviour (y^μ are now the SS coordinates)

$$(u^v)_{EF} = \frac{\partial v}{\partial y^\mu}(u^\mu)_{SS} = \frac{\partial v}{\partial t}(u^t)_{SS} = (u^t)_{SS} \ . \tag{19}$$

(a) Since $\dot{u}^\alpha = 0$ for a static oberver, the acceleration is

$$a^\alpha = \Gamma^\alpha_{\beta\gamma}u^\beta u^\gamma = \Gamma^\alpha_{vv}(u^v)^2 = f(r)^{-1}\Gamma^\alpha_{vv} \ . \tag{20}$$

Even though Christoffel symbols are non-tensorial in general, they do transform in a simple way under the very simple coordinate transformation between SS and EF coordinates. Nevertheless it is more convenient (and in any case a good exercise) to just calculate the relevant Christoffel symbols directly in EF coordinates rather than transforming them from the result in SS coordinates.

First of all, the Christoffel symbol $\Gamma_{\alpha vv}$ is rather obviously non-zero only for $\alpha = r$, and

$$\Gamma_{rvv} = -\tfrac{1}{2}g_{vv,r} = +\tfrac{1}{2}f'(r) = m/r^2 \ . \tag{21}$$

To determine Γ^α_{vv} we need the components of the inverse metric. Because the metric is not diagonal, this requires a bit of care. In matrix form, the (v,r)-components of the metric and its inverse are

$$(g_{\alpha\beta}) = \begin{pmatrix} -f & 1 \\ 1 & 0 \end{pmatrix} \quad , \quad (g^{\alpha\beta}) = \begin{pmatrix} 0 & 1 \\ 1 & f \end{pmatrix} \tag{22}$$

Therefore, there are two non-vanishing Christoffel symbols Γ^α_{vv}, namely

$$\Gamma^v_{vv} = \Gamma_{rvv} \quad , \quad \Gamma^r_{vv} = f(r)\Gamma_{rvv} \ , \tag{23}$$

and the acceleration vector is

$$(a^\alpha)_{EF} = (a^v = f(r)^{-1}m/r^2, a^r = m/r^2, 0, 0) \ . \tag{24}$$

Thus the (non-singular, “Newtonian”) r-component agrees with that of the acceleration in SS coordinates, but in addition in EF coordinates there is a v-component which is singular as $r \to 2m$.

(b) Alternatively, and more quickly, since a^α is a vector, one can obtain this result by transforming the result in SS coordinates to EF coordinates,

$$(a^\alpha)_{EF} = \frac{\partial x^\alpha}{\partial y^\mu}(a^\mu)_{SS} = \frac{\partial x^\alpha}{\partial r}(a^r)_{SS} \quad \Rightarrow \quad \begin{cases} (a^v)_{EF} = f(r)^{-1}m/r^2 \\ (a^r)_{EF} = m/r^2 \end{cases} \tag{25}$$

(c) Finally, the norm of the acceleration in EF coordinates is

$$g_{\alpha\beta}a^{\alpha}a^{\beta} = -f(r)(a^{v})^{2} + 2a^{v}a^{r} = f(r)^{-1}(m/r^{2})^{2} \ , \tag{26}$$

in complete agreement with the result in SS coordinates (as it should).