

## SOLUTIONS TO ASSIGNMENTS 06

### 1. SCALAR FIELDS IN A GRAVITATIONAL FIELD

The action is

$$S[\phi, g_{\alpha\beta}] = \int \sqrt{g} d^4x L \equiv -\frac{1}{2} \int \sqrt{g} d^4x (g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + m^2 \phi^2) \quad (1)$$

and the energy-momentum tensor is

$$T_{\alpha\beta} = \partial_\alpha \phi \partial_\beta \phi + g_{\alpha\beta} L \quad (2)$$

- (a) To show that the energy-momentum tensor is covariantly conserved we will use  $\partial_\mu \phi = \nabla_\mu \phi$ , the commutativity  $\nabla_\mu \nabla_\nu \phi = \nabla_\nu \nabla_\mu \phi$  of the covariant derivative on scalars, and the fact that  $\phi$  is a solution to the Klein-Gordon equation  $\nabla^\mu \nabla_\mu \phi = m^2 \phi$ , then the result follows:

$$\begin{aligned} \nabla^\mu T_{\mu\nu} &= \nabla^\mu (\partial_\mu \phi \partial_\nu \phi) + \nabla^\mu (g_{\mu\nu} L) \\ &= \nabla^\mu (\partial_\mu \phi \partial_\nu \phi) - \frac{1}{2} \nabla_\nu (\partial_\lambda \phi \partial^\lambda \phi + m^2 \phi^2) \\ &= \partial_\nu \phi \nabla^\mu \partial_\mu \phi + \partial_\mu \phi \nabla^\mu \partial_\nu \phi - \partial_\lambda \phi \nabla_\nu \partial^\lambda \phi - m^2 \phi \nabla_\nu \phi \\ &= \partial_\nu \phi m^2 \phi + \partial_\mu \phi \nabla^\mu \nabla_\nu \phi - \partial_\lambda \phi \nabla^\lambda \nabla_\nu \phi - m^2 \phi \nabla_\nu \phi = 0. \end{aligned} \quad (3)$$

- (b) The variation of the action with respect to the metric is

$$\delta S = \int d^4x (\delta(\sqrt{g})L + \sqrt{g}\delta L) = -\frac{1}{2} \int d^4x \sqrt{g} (g_{\mu\nu} L \delta g^{\mu\nu} - 2\delta L) \quad (4)$$

(valid for any Lagrangian  $L$ ). Using  $\delta(g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi) = (\delta g^{\mu\nu}) \partial_\mu \phi \partial_\nu \phi$ , one finds

$$\delta S = -\frac{1}{2} \int d^4x \sqrt{g} (g_{\mu\nu} L + \partial_\mu \phi \partial_\nu \phi) \delta g^{\mu\nu} = -\frac{1}{2} \int d^4x \sqrt{g} T_{\mu\nu} \delta g^{\mu\nu} \quad (5)$$

as claimed.

### 2. MAXWELL EQUATIONS IN A GRAVITATIONAL FIELD

The action is

$$S[A_\alpha, g_{\alpha\beta}] = \int \sqrt{g} d^4x L = -\frac{1}{4} \int \sqrt{g} d^4x F_{\alpha\beta} F^{\alpha\beta} \quad (6)$$

and the gauge-invariant and generally covariant energy momentum tensor is

$$T_{\alpha\beta} = F_{\alpha\gamma} F_\beta{}^\gamma - \frac{1}{4} g_{\alpha\beta} F_{\gamma\delta} F^{\gamma\delta} \quad (7)$$

- (a) We compute, using  $\nabla_\mu F^{\mu\lambda} = 0$ ,

$$\begin{aligned} \nabla_\mu T^{\mu\nu} &= \nabla_\mu (F_\lambda{}^\mu F^{\nu\lambda} - \frac{1}{4} g^{\mu\nu} F_{\lambda\rho} F^{\lambda\rho}) = F_\lambda{}^\mu \nabla_\mu F^{\nu\lambda} - \frac{1}{2} F_{\lambda\rho} \nabla^\nu F^{\lambda\rho} \\ &= F_{\mu\lambda} \left( \nabla^\mu F^{\nu\lambda} - \frac{1}{2} \nabla^\nu F^{\mu\lambda} \right) = \frac{1}{2} F_{\mu\lambda} \left( \nabla^\mu F^{\nu\lambda} - \nabla^\mu F^{\lambda\nu} - \nabla^\nu F^{\mu\lambda} \right) \end{aligned} \quad (8)$$

Using the anti-symmetry of  $F$ , one can write this as

$$\nabla_\mu T^{\mu\nu} = -\frac{1}{2} F_{\mu\lambda} \left( \nabla^\lambda F^{\nu\mu} + \nabla^\mu F^{\lambda\nu} + \nabla^\nu F^{\mu\lambda} \right) = 0 \quad (9)$$

- (b) For the metric variation of the action, we can also use the general formula (4). For the variation of the Lagrangian with respect to the metric, we note that

$$\delta(g^{\mu\lambda}g^{\nu\rho}F_{\mu\nu}F_{\lambda\rho}) = 2(\delta g^{\mu\lambda})g^{\nu\rho}F_{\mu\nu}F_{\lambda\rho} = 2(\delta g^{\mu\nu})g^{\lambda\rho}F_{\mu\lambda}F_{\nu\rho} = 2(\delta g^{\mu\nu})F_{\mu\lambda}F_{\nu}^{\lambda} \quad (10)$$

and therefore  $-2\delta L = (\delta g^{\mu\nu})F_{\mu\lambda}F_{\nu}^{\lambda}$ . Thus (7) follows.

#### PAINLEVÉ-GULLSTRAND COORDINATES FOR THE SCHWARZSCHILD SPACE-TIME

- (a) We make a coordinate transformation on the standard Schwarzschild metric with coordinates  $(t, r)$  defining a new coordinate  $T(t, r) = t + \psi(r)$ . This leads us to rewrite the metric with  $dT = dt + \psi' dr$  and we find

$$ds^2 = -f(r)dT^2 + 2f(r)\psi'dTdr + f(r)^{-1}(1 - f(r)^2\psi'^2)dr^2 + r^2d\Omega^2 \quad (11)$$

Choosing  $C(r) = f(r)\psi'$  gives the desired result and the function  $C(r)$  is completely arbitrary because  $\psi(r)$  is arbitrary.

- (b) For the Painlevé-Gullstrand coordinate we make a particular choice for  $C(r)$  namely  $C(r) = \sqrt{1 - f(r)}$  such that  $g_{rr} = f(r)^{-1}(1 - C(r)^2) = 1$ . We are thus left with the metric

$$ds^2 = -f(r)dT^2 + 2\sqrt{\frac{2m}{r}}dTdr + dr^2 + r^2d\Omega^2 \quad (12)$$

Now, with this new choice of coordinate we sees that any component  $g_{\mu\nu}$  of the metric stays finite for any value of  $r > 0$  (this was not the case at the beginning in the  $(t, r)$ -coordinates). In addition to that we also notice that the determinant of the metric is :

$$\det(g_{\mu\nu}) = (-f(r) - \frac{2m}{r})r^4 \sin^2(\theta) = -r^4 \sin^2(\theta)^2 \quad (13)$$

which is non-vanishing for any  $r > 0$  (with  $\theta \neq 0, \pi$  of course).

- (c) If we now make the choice  $C(r) = 1$ , then the metric becomes :

$$ds^2 = -f(r)dT^2 + 2dTdr + r^2d\Omega^2 \quad (14)$$

and if we rename  $T(t, r)$  to  $v(t, r)$ , then

$$ds^2 = -f(r)dv^2 + 2dvdr + r^2d\Omega^2 \quad (15)$$

is exactly the metric in the Eddington-Finkelstein coordinates.

We can also check explicitly that the coordinate transformation is indeed also the same. The particular choice  $C(r) = 1$  implies that  $\psi(r)$  is such that

$$C(r) = 1 \quad \Leftrightarrow \quad \psi'(r) = \frac{1}{f(r)} \quad \Leftrightarrow \quad \psi(r) = r^* + c, \quad (16)$$

where the constant  $c$  can be set to zero so that we have

$$T(t, r) = t + \psi(r) = t + r^* = v(t, r) \quad (17)$$

as we should.