

# SOLUTIONS TO ASSIGNMENTS 07

## 1. RIEMANN CURVATURE TENSOR AND DIFFERENTIAL IDENTITIES

- (a) Writing  $\circlearrowleft$  for the cyclic permutations in  $(\alpha, \beta, \gamma)$  and then using the cyclic symmetry  $R^\rho_{\alpha\beta\gamma} + \circlearrowleft = 0$ , we have (for any  $V$ )

$$\begin{aligned}
 0 &= [\nabla_\alpha, [\nabla_\beta, \nabla_\gamma]]V^\lambda + \circlearrowleft \\
 &= \nabla_\alpha(R^\lambda_{\rho\beta\gamma}V^\rho) - R^\lambda_{\rho\beta\gamma}\nabla_\alpha V^\rho + R^\rho_{\alpha\beta\gamma}\nabla_\rho V^\lambda + \circlearrowleft \\
 &= (\nabla_\alpha R^\lambda_{\rho\beta\gamma})V^\rho + R^\rho_{\alpha\beta\gamma}\nabla_\rho V^\lambda + \circlearrowleft = (\nabla_\alpha R^\lambda_{\rho\beta\gamma})V^\rho + \circlearrowleft \\
 &\Leftrightarrow \nabla_\alpha R_{\lambda\rho\beta\gamma} + \circlearrowleft = 0 .
 \end{aligned} \tag{1}$$

- (b) Contracting the Bianchi identity over the indices  $(\mu, \beta)$  and  $(\nu, \alpha)$  one finds

$$\begin{aligned}
 0 &= g^{\mu\alpha}g^{\nu\beta}[\nabla_\alpha R_{\mu\nu\beta\gamma} + \nabla_\beta R_{\mu\nu\gamma\alpha} + \nabla_\gamma R_{\mu\nu\alpha\beta}] \\
 &= \nabla_\alpha R^{\alpha\beta}_{\beta\gamma} + \nabla_\beta R^{\alpha\beta}_{\gamma\alpha} + \nabla_\gamma R^{\alpha\beta}_{\alpha\beta} \\
 &= -\nabla_\alpha R^\alpha_\gamma - \nabla_\beta R^\beta_\gamma + \nabla_\gamma R = -\nabla_\alpha [2R^\alpha_\gamma - \delta^\alpha_\gamma R] \\
 &\Leftrightarrow \nabla^\alpha G_{\alpha\beta} \equiv \nabla^\alpha (R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R) = 0
 \end{aligned} \tag{2}$$

- (c) The identity  $[\nabla_\alpha, \nabla_\beta]T^{\gamma\delta} = R^\gamma_{\epsilon\alpha\beta}T^{\epsilon\delta} + R^\delta_{\epsilon\alpha\beta}T^{\gamma\epsilon}$  implies

$$[\nabla_\alpha, \nabla_\beta]T^{\alpha\beta} = R^\alpha_{\epsilon\alpha\beta}T^{\epsilon\beta} + R^\beta_{\epsilon\alpha\beta}T^{\alpha\epsilon} = R_{\epsilon\beta}T^{\epsilon\beta} - R_{\epsilon\alpha}T^{\alpha\epsilon} = R_{\alpha\beta}(T^{\alpha\beta} - T^{\beta\alpha}) = 0 \tag{3}$$

because  $R_{\alpha\beta} = R_{\beta\alpha}$  is symmetric.

- (d) Because of this identity and the fact that  $F^{\alpha\beta} = -F^{\beta\alpha}$ , one has

$$-\nabla_\beta J^\beta = \nabla_\beta \nabla_\alpha F^{\alpha\beta} = \frac{1}{2}[\nabla_\beta, \nabla_\alpha]F^{\alpha\beta} = 0 . \tag{4}$$

## 2. CURVATURE OF A CLASS OF 2-DIMENSIONAL METRICS

The Lagrangian  $\mathcal{L} = (\dot{x}^2 + f(x)^2\dot{\phi}^2)/2$  implies the Euler-Lagrange equations  $\ddot{x} - ff'\dot{\phi}^2 = 0$  and  $\ddot{\phi} + 2(f'/f)\dot{x}\dot{\phi} = 0$ , where  $f' \equiv df/dx$ . From this one reads off the non-zero Christoffel symbols  $\Gamma^x_{\phi\phi} = -ff'$ ,  $\Gamma^\phi_{x\phi} = \Gamma^\phi_{\phi x} = f'/f$ , and thus

$$\begin{aligned}
 R_{x\phi x\phi} &= R^x_{\phi x\phi} = \partial_x \Gamma^x_{\phi\phi} - \partial_\phi \Gamma^x_{\phi x} + \Gamma^x_{x\alpha} \Gamma^\alpha_{\phi\phi} - \Gamma^x_{\phi\alpha} \Gamma^\alpha_{\phi x} \\
 &= \partial_x \Gamma^x_{\phi\phi} - \Gamma^x_{\phi\phi} \Gamma^\phi_{\phi x} = -(f')^2 - ff'' + (ff')(f'/f) = -ff'' \\
 \Rightarrow R^\phi_{x\phi x} &= g^{\phi\alpha} R_{\alpha x\phi x} = g^{\phi\phi} R_{\phi x\phi x} = R_{\phi x\phi x}/f^2 = R_{x\phi x\phi}/f^2 = -f''/f \\
 \Rightarrow R_{xx} &= R^\alpha_{x\alpha x} = R^\phi_{x\phi x} = -f''/f , \quad R_{\phi\phi} = R^x_{\phi x\phi} = -ff'' , \quad R_{x\phi} = 0 \\
 \Rightarrow R_{\alpha\beta} &= -(f''/f)g_{\alpha\beta} \Rightarrow R = -2f''/f
 \end{aligned} \tag{5}$$