1. ON THE MAXWELL EQUATIONS IN CURVED SPACE-TIME

The action is

\[ S[A_\alpha, g_{\alpha\beta}] = \int \sqrt{g} d^4x \, L = -\frac{1}{4} \int \sqrt{g} d^4x F_{\alpha\beta} F^{\alpha\beta} \]  

(1)

and the gauge-invariant and generally covariant energy momentum tensor is

\[ T_{\alpha\beta} = F_{\alpha\gamma} F^{\gamma\beta} - \frac{1}{4} g_{\alpha\beta} F^{\gamma\delta} F_{\gamma\delta} \]  

(2)

(a) The variation of the action with respect to the gauge field is

\[ \delta S = -\int \sqrt{g} d^4x \left( \partial_\mu \delta A_\nu \right) F^{\mu\nu} \]

\[ = -\int d^4x \partial_\mu (\sqrt{g} \delta A_\nu F^{\mu\nu}) + \int d^4x \partial_\mu (\sqrt{g} F^{\mu\nu}) \delta A_\nu \]  

(3)

where in the first line we have used the fact that there are 4 identical contributions to the variation. Then we note that the first term in the second line is a boundary term and vanishes because the variation vanishes on the boundary. Now using \( \sqrt{g} \nabla_\mu F^{\mu\nu} = \partial_\mu (\sqrt{g} F^{\mu\nu}) \) we are left with

\[ \delta S = \int \sqrt{g} d^4x \left( \nabla_\mu F^{\mu\nu} \right) \delta A_\nu \]  

(4)

which gives the equations of motion.

(b) We compute:

\[ \nabla_\mu T^{\mu\nu} = \nabla_\mu (F^{\mu\nu} F^{\nu\lambda} - \frac{1}{3} g^{\mu\nu} F_{\lambda\rho} F^{\lambda\rho}) \]

\[ = (\nabla_\mu F^{\mu\nu} F^{\nu\lambda} + F^{\mu\nu} \nabla_\mu F^{\nu\lambda} - \frac{1}{2} F_{\lambda\rho} \nabla^{\nu} F^{\lambda\rho}) \]

\[ = -J_\lambda F^{\nu\lambda} + F_{\mu\lambda} \left( \nabla^{\nu} F^{\mu\lambda} - \frac{1}{2} \nabla^{\nu} F_{\mu\lambda} \right) \]

\[ = J_\lambda F^{\nu\lambda} + \frac{1}{2} F_{\mu\lambda} \left( \nabla^{\nu} F^{\mu\lambda} - \nabla^{\lambda} F^{\mu\nu} - \nabla^{\nu} F^{\mu\lambda} \right) \]  

(5)

Then we rewrite the term \( \frac{1}{2} F_{\mu\lambda} \nabla^{\nu} F^{\mu\lambda} \) in a different way by relabeling the indices and using the anti-symmetry of \( F_{\mu\nu} \):

\[ \frac{1}{2} F_{\mu\lambda} \nabla^{\nu} F^{\mu\lambda} = \frac{1}{2} F_{\lambda\mu} \nabla^{\nu} F^{\nu\mu} = -\frac{1}{2} F_{\mu\lambda} \nabla^{\nu} F^{\nu\mu} \]  

(6)

so that we can now use \( \nabla_{[\lambda} F_{\mu\nu]} = 0 \) to have at the end:

\[ \nabla_\mu T^{\mu\nu} = J_\lambda F^{\lambda\nu} - \frac{1}{2} F_{\mu\lambda} \left( \nabla^{\lambda} F^{\mu\nu} + \nabla^{\nu} F^{\mu\lambda} + \nabla^{\nu} F^{\mu\lambda} \right) \]

\[ = J_\lambda F^{\lambda\nu} \]  

(7)
(c) For the metric variation of the action, we can also use the general formula

$$\delta S = \int d^4x (\delta(\sqrt{g})L + \sqrt{g}\delta L) = -\frac{1}{2} \int d^4x \sqrt{g} (g_{\mu\nu} \delta g^{\mu\nu} - 2\delta L)$$

(8)

(valid for any Lagrangian $L$). For the variation of the Lagrangian with respect to the metric, we note that

$$\delta(g^{\mu\lambda} g^{\nu\rho} F_{\mu\nu} F_{\lambda\rho}) = 2(\delta g^{\mu\lambda}) g^{\nu\rho} F_{\mu\nu} F_{\lambda\rho} = 2(\delta g^{\mu\nu}) g^{\lambda\rho} F_{\mu\nu} F_{\lambda\rho} = 2(\delta g^{\mu\nu}) F_{\mu\lambda} F^{\lambda}_\nu$$

(9)

and therefore $-2\delta L = (\delta g^{\mu\nu}) F_{\mu\lambda} F^{\lambda}_\nu$. Putting the pieces together one gets (2).

2. Kruskal Coordinates for the Schwarzschild Space-Time: Solution I

(Direct calculation using the coordinate transformation)

To compute the Schwarzschild metric in the new $(X,T)$-coordinate, it’s useful to consider the two expression $t(X,T)$ and $r^*(X,T)$. To find these we first rewrite $(X,T)$ as :

$$X = e^{r^*/4m} \cosh(t/4m)$$

$$T = e^{r^*/4m} \sinh(t/4m) .$$

(10)

This leads in particular to :

$$X^2 - T^2 = e^{r^*/2m} = e^{r/2m} \left( \frac{r}{2m} - 1 \right) = r F(r) \frac{e^{r/2m}}{2m}$$

(11)

which is a way to express $r$ implicitly (we have defined $F := \frac{\partial r}{\partial r^*} = 1 - \frac{2m}{r}$ for later use). Now, from (6) it also follows that :

$$t = 4m \text{ atanh} \left( \frac{T}{X} \right)$$

$$r^* = 2m \log \left( X^2 - T^2 \right) .$$

(12)

and this allows us to compute the partial derivative we will need :

$$\frac{\partial t}{\partial T} = \frac{4mX}{X^2 - T^2} \quad \frac{\partial r}{\partial T} = \frac{\partial r}{\partial r^*} \frac{\partial r^*}{\partial T} = F \frac{4mT}{T^2 - X^2}$$

$$\frac{\partial t}{\partial X} = \frac{4mT}{T^2 - X^2} \quad \frac{\partial r}{\partial X} = \frac{\partial r}{\partial r^*} \frac{\partial r^*}{\partial X} = F \frac{4mX}{X^2 - T^2}$$

(13)

Then it is straightforward to compute the Schwarzschild metric starting from the
old \((t,r)\)-coordinate and we get:

\[
\begin{align*}
\text{ds}^2 &= -F dt^2 + F^{-1} dr^2 + r^2 d\Omega^2 \\
&= -F \left( \frac{\partial t}{\partial T} dT + \frac{\partial t}{\partial X} dX \right)^2 + F^{-1} \left( \frac{\partial r}{\partial T} dT + \frac{\partial r}{\partial X} dX \right)^2 + r^2 d\Omega^2 \\
&= \frac{16m^2 F}{(X^2 - T^2)^2} \left[ - (X dT - T dX)^2 + (-T dT + X dX)^2 \right] + r^2 d\Omega^2 \\
&= \frac{16m^2 F}{(X^2 - T^2)} \left[ -dT^2 + dX^2 \right] + r^2 d\Omega^2 \\
&= \frac{32m^3}{r} e^{-r/2m} \left[ -dT^2 + dX^2 \right] + r^2 d\Omega^2 \tag{14}
\end{align*}
\]

where in the last step we have used (8).

2. **Kruskal Coordinates for the Schwarzschild Space-Time: Solution II**

(Massaging the metric into a convenient form)

Write the Schwarzschild metric as

\[
\begin{align*}
\text{ds}^2 &= (1 - 2m/r) (-dt^2 + dr^2) + r^2 d\Omega^2 \\
&= (1 - 2m/r) [-du \, dv] + r(u,v)^2 d\Omega^2 \tag{15}
\end{align*}
\]

where \( r^* = r + 2m \log(r/2m - 1) \) is the tortoise coordinate, and \( v = t + r^* \); \( u = t - r^* \) are the “advanced” and “retarded” Eddington-Finkelstein coordinates.

Now note that

\[
\frac{v - u}{4m} = \frac{r}{2m} + \log \left( \frac{r}{2m} - 1 \right), \tag{16}
\]

so that

\[
1 - \frac{2m}{r} = \frac{2m}{r} \left( \frac{r}{2m} - 1 \right) = \frac{2m}{r} e^{-r/2m} e(v - u)/4m. \tag{17}
\]

Thus the metric is

\[
\begin{align*}
\text{ds}^2 &= \frac{2m}{r} e^{-r/2m} \left( e^{v/4m} \, dv \right) \left( -e^{-u/4m} \, du \right) + r(u,v)^2 d\Omega^2 \tag{18}
\end{align*}
\]

Therefore it is natural to introduce

\[
V = e^{v/4m}, \quad U = -e^{-u/4m}, \tag{19}
\]

and \( T \) and \( X \) via \( V = T + X, U = T - X \), so that

\[
\begin{align*}
\text{ds}^2 &= -\frac{32m^3}{r} e^{-r/2m} dU \, dV + r(u,v)^2 d\Omega^2 \\
&= \frac{32m^3}{r} e^{-r/2m} \left[ -dT^2 + dX^2 \right] + r(T, X)^2 d\Omega^2 \tag{20}
\end{align*}
\]

Moreover, equation (17) implies that

\[
\left( \frac{r}{2m} - 1 \right) e^{r/2m} = e^{(v - u)/4m} = -UV = X^2 - T^2, \tag{21}
\]

so that \( r = 2m \Leftrightarrow X = \pm T \) and \( r = 0 \Leftrightarrow X^2 - T^2 = -1. \)
3. Painlevé-Gullstrand Coordinates for the Schwarzschild Space-Time

(a) We make a coordinate transformation on the standard Schwarzschild metric with coordinates \((t, r)\) defining a new coordinate \(T(t, r) = t + \psi(r)\). This leads us to rewrite the metric with \(dT = dt + \psi' dr\) and we find:

\[
ds^2 = -f(r) dT^2 + 2f(r)\psi' dT dr + f(r)^{-1}(1 - f(r)^{2}\psi'^2) dr^2 + r^2 d\Omega^2
\]  

(22)

Choosing \(C(r) = f(r)\psi'\) gives the desired result and the function \(C(r)\) is completely arbitrary because \(\psi(r)\) is arbitrary.

(b) In the Painlevé-Gullstrand coordinate we make a particular choice for \(C(r)\) namely \(C(r) = \sqrt{1 - f(r)}\) such that \(g_{rr} = f(r)^{-1}(1 - C(r)^2) = 1\). We are thus left with the metric:

\[
ds^2 = -f(r) dT^2 + 2\sqrt{\frac{2m}{r}} dT dr + dr^2 + r^2 d\Omega^2
\]  

(23)

Now, with this new choice of coordinate we see that any component \(g_{\mu\nu}\) of the metric stays finite for any value of \(r > 0\) (it was not the case at the beginning in the \((t, r)\)-coordinate). In addition to that we also notice that the determinant of the metric is:

\[
det(g_{\mu\nu}) = (-f(r) - \frac{2m}{r}) r^4 \sin(\theta)^2 = -r^4 \sin(\theta)^2
\]  

(24)

which is non-vanishing for any \(r > 0\) (with \(\theta \neq \pm 0, \pi\) of course).

(c) If we now make the choice \(C(r) = 1\), then the metric becomes:

\[
ds^2 = -f(r) dT^2 + 2dT dr + r^2 d\Omega^2
\]  

(25)

and if we rename \(T(t, r) \to v(t, r)\), then:

\[
ds^2 = -f(r) dv^2 + 2dv dr + r^2 d\Omega^2
\]  

(26)

which is exactly the metric in the Eddington-Finkelstein coordinates. We can check explicitly that the transformation is indeed the same. The particular choice \(C(r) = 1\) implies that \(\psi(r)\) is such that \(\psi' = \frac{1}{f(r)}\). Thus \(\psi = r^* + c\) where the constant can be set to zero so that we have \(T(t, r) = t + \psi = t + r^* = v(t, r)\).