1. **Explicit Expression for the Riemann Curvature Tensor**

\[ [\nabla_\mu, \nabla_\nu] V^\lambda = \nabla_\mu \left[ \partial_\nu V^\lambda + \Gamma^\lambda_{\nu\rho} V^\rho \right] - (\mu \leftrightarrow \nu) \]

\[ = \partial_\mu \partial_\nu V^\lambda + \partial_\mu (\Gamma^\lambda_{\nu\rho} V^\rho) + \Gamma^\lambda_{\mu\rho} \partial_\nu V^\rho - \Gamma^\lambda_{\mu\nu} \partial_\rho V^\lambda + \Gamma^\lambda_{\mu\sigma} \Gamma^\sigma_{\nu\rho} V^\rho - \Gamma^\lambda_{\mu\sigma} \Gamma^\sigma_{\nu\rho} V^\rho - (\mu \leftrightarrow \nu) \]

\[ = \partial_\mu (\Gamma^\lambda_{\nu\rho} V^\rho) + \Gamma^\lambda_{\mu\rho} \partial_\nu V^\rho + \Gamma^\lambda_{\mu\sigma} \Gamma^\sigma_{\nu\rho} V^\rho - (\mu \leftrightarrow \nu) \]

\[ = (\partial_\mu \Gamma^\lambda_{\nu\rho}) V^\rho + \Gamma^\lambda_{\nu\rho} \partial_\mu V^\rho + \Gamma^\lambda_{\mu\sigma} \Gamma^\sigma_{\nu\rho} V^\rho - (\mu \leftrightarrow \nu) \]

\[ = \left( \partial_\mu \Gamma^\lambda_{\nu\rho} - \partial_\nu \Gamma^\lambda_{\mu\rho} + \Gamma^\lambda_{\mu\sigma} \Gamma^\sigma_{\nu\rho} - \Gamma^\lambda_{\mu\sigma} \Gamma^\sigma_{\nu\rho} \right) V^\rho = R^\lambda_{\mu\nu\rho} V^\rho \]

where in the third equality we dropped all the \((\mu, \nu)\)-symmetric terms killed by the subtraction with the indices exchanged.

2. **Properties of the Riemann Curvature Tensor**

(a) We show that the fourth symmetry follows from (I),(II) and (III):

\[ R_{\alpha\beta\gamma\delta} = - (R_{\alpha\delta\beta\gamma} + R_{\alpha\gamma\delta\beta}) = R_{\gamma\alpha\delta\beta} + R_{\delta\alpha\gamma\beta} \]

\[ = -(R_{\gamma\delta\beta\alpha} + R_{\gamma\beta\delta\alpha}) - (R_{\delta\beta\gamma\alpha} + R_{\delta\gamma\beta\alpha}) \]

\[ = 2R_{\gamma\delta\alpha\beta} + R_{\beta\gamma\alpha\delta} + R_{\beta\delta\gamma\alpha} \]

\[ = 2R_{\gamma\delta\alpha\beta} - R_{\beta\alpha\delta\gamma} = 2R_{\gamma\delta\alpha\beta} - R_{\alpha\beta\gamma\delta} \]

\[ \Rightarrow R_{\alpha\beta\gamma\delta} = R_{\gamma\delta\alpha\beta} \]

(b) From (a) we directly deduce the symmetry of the Ricci tensor :

\[ R_{\mu\nu} = R^\rho_{\mu\rho\nu} = R^\rho_{\nu\mu\rho} = R^\rho_{\nu\rho\mu} = R^\rho_{\nu\mu\rho} \]

(c) Writing \(\circ\) for the cyclic permutations in \((\alpha, \beta, \gamma)\) and then using the third symmetry : \(R^\rho_{\alpha\beta\gamma} + \circ = 0\), we have :

\[ [\nabla_\alpha, [\nabla_\beta, \nabla_\gamma]] V^\lambda + \circ = \nabla_\alpha \left( R^\lambda_{\beta\gamma} \nabla_\rho V^\rho \right) - R^\lambda_{\rho\beta\gamma} \nabla_\alpha V^\rho + R^\rho_{\alpha\beta\gamma} \nabla_\rho V^\lambda + \circ \]

\[ = \nabla_\alpha \left( R^\lambda_{\beta\gamma} \nabla_\rho V^\rho \right) + R^\rho_{\alpha\beta\gamma} \nabla_\rho V^\lambda + \circ \]

\[ = \nabla_\alpha \left( R^\lambda_{\beta\gamma} \nabla_\rho V^\rho \right) + \circ \]

\[ = g^{\lambda\mu} [\nabla_\alpha R_{\mu\nu\beta\gamma} + \circ] V^\nu = 0 \]

which gives the desired result.

(d) Contracting the Bianchi identity over the indices \((\mu, \beta)\) and \((\nu, \alpha)\) one finds :

\[ g^{\nu\alpha} g^{\mu\beta} \left[ \nabla_\alpha R_{\mu\nu\beta\gamma} + \circ \right] = g^{\nu\alpha} g^{\mu\beta} \left[ \nabla_\alpha R_{\mu\nu\beta\gamma} + \nabla_\beta R_{\mu\nu\gamma\alpha} + \nabla_\gamma R_{\mu\nu\alpha\beta} \right] \]

\[ = \nabla_\alpha R_{\alpha\beta\gamma} + \nabla_\beta R_{\alpha\gamma\delta} + \nabla_\gamma R_{\alpha\beta\delta} \]

\[ = - \nabla_\alpha R_{\alpha\gamma} - \nabla_\beta R_{\beta\gamma} + \nabla_\gamma R \]

\[ = - \nabla_\alpha \left[ 2R^\alpha_{\gamma} - \delta^\alpha_{\gamma} R \right] = 0 \]
And defining the Einstein tensor as \(G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R\), we see that the contracted Bianchi identity (4) is equivalent to \(\nabla^\alpha G_{\alpha\beta} = 0\) because:

\[
\nabla^\alpha G_{\alpha\beta} = \nabla^\alpha (R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R) = \frac{1}{2} \nabla_\alpha (2R_{\gamma\beta} - g^{\alpha\beta}R) \tag{6}
\]

so that \(\nabla^\alpha G_{\alpha\beta} = 0 \iff \nabla_\alpha \left[ 2R_{\gamma\beta} - \delta^{\alpha\beta}R \right] = 0\) where we have use the fact that \(g^{\alpha\beta} = g^{\alpha\lambda}g_{\lambda\beta} = \delta^{\alpha\beta}\) simply because \(g^{\alpha\lambda}\) is the inverse of \(g_{\alpha\lambda}\).

3. The Geodesic Deviation Equation (section 8.3)

(a) This is obvious (I hope) since from expansion of the displaced equation to first order one gets

\[
\Gamma^\mu_{\nu\lambda}(x + \delta x) \frac{d}{d\tau}(x^\nu + \delta x^\nu) \frac{d}{d\tau}(x^\lambda + \delta x^\lambda) = \partial^\rho \Gamma^\mu_{\nu\lambda}(x) \delta x^\rho \frac{d}{d\tau} x^\nu \frac{d}{d\tau} x^\lambda + 2\Gamma^\mu_{\nu\lambda}(x) \frac{d}{d\tau} x^\nu \frac{d}{d\tau} \delta x^\lambda \tag{7}
\]

(the symmetry of the Christoffel symbols accounting for the factor of 2).

(b) Starting from

\[
\frac{D}{D\tau} \delta x^\mu = \frac{d}{d\tau} \delta x^\mu + \Gamma^\mu_{\nu\lambda} \frac{d}{d\tau} x^\nu \delta x^\lambda \tag{8}
\]

one can calculate the 2nd derivative \(D^2 \delta x^\mu / D\tau^2\),

\[
\frac{D^2}{D\tau^2} \dot{x}^\mu = \frac{d}{d\tau} \left[ \delta \dot{x}^\mu + \Gamma^\mu_{\nu\rho} \dot{x}^\nu \delta x^\rho \right] + \Gamma^\mu_{\alpha\beta} \dot{x}^\alpha \left[ \delta \dot{x}^\beta + \Gamma^\beta_{\nu\rho} \dot{x}^\nu \delta x^\rho \right] = \delta \ddot{x}^\mu + (\partial^\lambda \Gamma^\mu_{\nu\rho}) \dot{x}^\nu \dot{x}^\lambda \delta x^\rho + \Gamma^\mu_{\nu\rho} \dot{x}^\nu \delta x^\rho + 2\Gamma^\mu_{\nu\rho} \dot{x}^\nu \delta \dot{x}^\rho + \Gamma^\mu_{\lambda\beta} \Gamma^\beta_{\nu\rho} \dot{x}^\lambda \dot{x}^\nu \delta x^\rho \tag{9}
\]

Subtracting from this the term \(R^\mu_{\nu\rho\lambda} \dot{x}^\nu \dot{x}^\lambda \delta x^\rho\) and using the geodesic equation to eliminate \(\ddot{x}^\nu\), one finds the equation

\[
\frac{d^2}{d\tau^2} \delta x^\mu + 2\Gamma^\mu_{\nu\lambda}(x) \frac{d}{d\tau} x^\nu \frac{d}{d\tau} x^\lambda + \partial^\nu \Gamma^\mu_{\nu\lambda}(x) \delta x^\rho \frac{d}{d\tau} x^\nu \frac{d}{d\tau} x^\lambda = 0 \tag{10}
\]