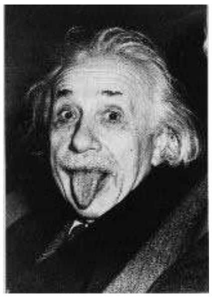
Formelsammlung Klassische Feldtheorie



1 Lorentz-Transformationen

1. Lorentz-Transformationen in $(1+1)$ Dimensionen
   (Koordinaten (t, x^1) oder $\{x^\alpha\} = \{x^0 = ct, x^1\}$)

   (a) Ausgangspunkt

   $$\begin{aligned} \bar{t} &= \frac{1}{\sqrt{1 - v^2/c^2}}(t - (v/c^2)x^1) \\ \bar{x}^1 &= \frac{1}{\sqrt{1 - v^2/c^2}}(x^1 - vt) \end{aligned} \tag{1}$$

   (b) die dimensionslosen Parameter $\beta = \beta(v)$ und $\gamma = \gamma(v)$:

   $$\beta(v) = v/c \qquad \gamma(v) = (1 - \beta(v)^2)^{-1/2} \tag{2}$$

   (c) Lorentz-Transformation ausgedrückt durch $\beta(v)$ und $\gamma(v)$:

   $$\begin{aligned} \bar{x}^0 &= \gamma(v)(x^0 - \beta(v)x^1) \\ \bar{x}^1 &= \gamma(v)(x^1 - \beta(v)x^0) \end{aligned} \tag{3}$$

   (d) Rapidität $\alpha = \alpha(v)$:

   $$\gamma(v)^2 - \gamma(v)^2\beta(v)^2 = 1 \quad \Rightarrow \quad \exists\, \alpha(v) : \quad \begin{aligned} \gamma(v) &= \cosh\alpha(v) \\ \gamma(v)\beta(v) &= \sinh\alpha(v) \\ \beta(v) &= \tanh\alpha(v) \end{aligned} \tag{4}$$

   (e) Lorentz-Transformation ausgedrückt durch $\alpha(v)$:

   $$\begin{aligned} \bar{x}^0 &= \cosh\alpha\, x^0 - \sinh\alpha\, x^1 \\ \bar{x}^1 &= -\sinh\alpha\, x^0 + \cosh\alpha\, x^1 \end{aligned} \tag{5}$$

   (f) In Matrixform (hyperbolische Rotation)

   $$\begin{pmatrix} \bar{x}^0 \\ \bar{x}^1 \end{pmatrix} = \underbrace{\begin{pmatrix} \cosh\alpha & -\sinh\alpha \\ -\sinh\alpha & \cosh\alpha \end{pmatrix}}_{L(\alpha)} \begin{pmatrix} x^0 \\ x^1 \end{pmatrix} \tag{6}$$

   (g) In Lichtkegelkoordinaten $x^\pm = x^0 \pm x^1$:

   $$\bar{x}^\pm = \mathrm{e}^{\mp\alpha} x^\pm \qquad \mathrm{e}^{-\alpha} = \sqrt{\frac{1 - v/c}{1 + v/c}} \tag{7}$$

2. Einige Eigenschaften der Lorentz-Transformationen

   (a) Invarianz des (Minkowski-) Abstandsquadrats

$$-(\bar{x}^0)^2 + (\bar{x}^1)^2 = -(x^0)^2 + (x^1)^2 \tag{8}$$

   (b) Additivität der Rapidität

$$L(\alpha_1)L(\alpha_2) = L(\alpha_1 + \alpha_2) \tag{9}$$

   (c) relativistische Geschwindigkeitsadditions-Formel

$$\alpha_3 \equiv \alpha_1 + \alpha_2 \quad , \quad \beta_3 = \tanh \alpha_3 \qquad \Rightarrow \qquad v_3 = \frac{v_1 + v_2}{1 + \frac{v_1 v_2}{c^2}} \tag{10}$$

   (d) relativistischer Dopplereffekt

$$\mathrm{e}^{-i\omega x^-} = \mathrm{e}^{-i\omega(ct - x^1)} \stackrel{!}{=} \mathrm{e}^{-i\bar{\omega}\bar{x}^-} \quad \Rightarrow \quad \bar{\omega} = \mathrm{e}^{-\alpha}\omega \tag{11}$$

3. Invariante Charakterisierung von Lorentz-Transformationen
   ($\{x^\alpha\} = \{x^0, x^1, x^2, x^3\}$ oder allgemeiner $\{x^\alpha\} = \{x^0, x^1, \ldots, x^d\}$)

   (a) Minkowski-Metrik

$$\eta_{\alpha\beta} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{12}$$

   oder allgemeiner

$$\eta_{\alpha\beta} = \mathsf{diag}(-1, \underbrace{+1, \ldots, +1}_{d \text{ mal}}) \tag{13}$$

   (b) Minkowski-Abstandsquadrat

$$\eta_{\alpha\beta} x^\alpha x^\beta = -(x^0)^2 + (x^1)^2 + \ldots + (x^d)^2 \tag{14}$$

   (c) d'Alembert (Wellen-) Operator $\Box$

$$\Box = -\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \sum_{i=1}^{d} \frac{\partial^2}{(\partial x^i)^2} = \eta^{\alpha\beta}\frac{\partial}{\partial x^\alpha}\frac{\partial}{\partial x^\beta} \qquad (\eta^{\alpha\beta}\eta_{\beta\gamma} = \delta^\alpha_\gamma \quad \text{inverse Metrik}) \tag{15}$$

   (d) Lorentz-Transformationen = Lineare Transformationen $\bar{x}^\alpha = L^\alpha_{\ \beta}\, x^\beta$
   die Minkowski Abstandsquadrat invariant lassen:

$$\eta_{\alpha\beta}\bar{x}^\alpha\bar{x}^\beta = \eta_{\alpha\beta}x^\alpha x^\beta \quad \Leftrightarrow \quad \eta_{\alpha\beta}L^\alpha_{\ \gamma}L^\beta_{\ \delta} = \eta_{\gamma\delta} \quad \Leftrightarrow \quad L^T\eta L = \eta \tag{16}$$

   (e) Äquivalente Charakterisierung:
   Invarianz des d'Alembert (Wellen-) Operators

$$\eta^{\alpha\beta}\frac{\partial}{\partial\bar{x}^\alpha}\frac{\partial}{\partial\bar{x}^\beta} = \eta^{\alpha\beta}\frac{\partial}{\partial x^\alpha}\frac{\partial}{\partial x^\beta} \tag{17}$$