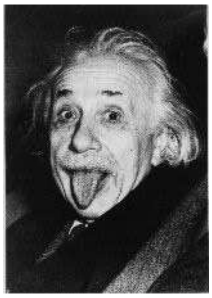
Formelsammlung Klassische Feldtheorie



5 Kovariante Formulierung der Maxwell-Theorie

1. Nicht-kovariante Formulierung

   (a) Homogene Gleichungen

   $$\vec{\nabla}.\vec{B} = 0 \quad , \quad \vec{\nabla} \times \vec{E} + \partial_t \vec{B} = 0 \tag{1}$$

   (b) Inhomogene Gleichungen

   $$\vec{\nabla}.\vec{E} = \rho/\epsilon_0 \quad , \quad \vec{\nabla} \times \vec{B} - \frac{1}{c^2}\partial_t \vec{E} = \mu_0 \vec{J} \tag{2}$$

   (c) Kontinuitätsgleichung

   $$\partial_t \rho + \vec{\nabla}.\vec{J} = 0 \tag{3}$$

   (d) Potentiale

   $$\vec{B} = \vec{\nabla} \times \vec{A} \quad , \quad \vec{E} = -\vec{\nabla}\phi - \partial_t \vec{A} \tag{4}$$

   (e) Eichtransformationen

   $$\vec{A} \to \vec{A} + \vec{\nabla}\Psi \quad , \quad \phi \to \phi - \partial_t \Psi \quad \Rightarrow \quad \vec{E} \to \vec{E} \quad , \quad \vec{B} \to \vec{B} \ . \tag{5}$$

2. 4er-Strom, 4er-Potential und Eichtransformationen

   $$J^\alpha = (c\rho, \vec{J}) \quad , \quad A_\alpha = (-\phi/c, \vec{A}) \quad , \quad A_\alpha \to A_\alpha + \partial_\alpha \Psi \tag{6}$$

3. Kontinuitätsgleichung

   $$\partial_t \rho + \vec{\nabla}.\vec{J} = 0 \quad \Leftrightarrow \quad \partial_\alpha J^\alpha = 0 \tag{7}$$

4. Maxwell Feldstärketensor

   $$F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha \ : \quad F_{0k} = -F_{k0} = -E_k/c \ , \ F_{ik} = \epsilon_{ik\ell}\, B_\ell \tag{8}$$

   $$(F_{\alpha\beta}) = \begin{pmatrix} 0 & -E_1/c & -E_2/c & -E_3/c \\ +E_1/c & 0 & +B_3 & -B_2 \\ +E_2/c & -B_3 & 0 & +B_1 \\ +E_3/c & +B_2 & -B_1 & 0 \end{pmatrix} \tag{9}$$

   $$(F^{\alpha\beta}) = \begin{pmatrix} 0 & +E_1/c & +E_2/c & +E_3/c \\ -E_1/c & 0 & +B_3 & -B_2 \\ -E_2/c & -B_3 & 0 & +B_1 \\ -E_3/c & +B_2 & -B_1 & 0 \end{pmatrix} \tag{10}$$

5. Dualer Feldstärketensor

$$\tilde{F}^{\alpha\beta} = \tfrac{1}{2}\epsilon^{\alpha\beta\gamma\delta}F_{\gamma\delta} \;:\quad \tilde{F}^{0k} = -\tilde{F}^{k0} = -B_k\;,\; \tilde{F}^{ik} = \epsilon_{ik\ell}\,E_\ell/c \tag{11}$$

$$(\tilde{F}^{\alpha\beta}) = \begin{pmatrix} 0 & -B_1 & -B_2 & -B_3 \\ +B_1 & 0 & +E_3/c & -E_2/c \\ +B_2 & -E_3/c & 0 & +E_1/c \\ +B_3 & +E_2/c & -E_1/c & 0 \end{pmatrix} \tag{12}$$

$$(\tilde{F}_{\alpha\beta}) = \begin{pmatrix} 0 & +B_1 & +B_2 & +B_3 \\ -B_1 & 0 & +E_3/c & -E_2/c \\ -B_2 & -E_3/c & 0 & +E_1/c \\ -B_3 & +E_2/c & -E_1/c & 0 \end{pmatrix} \tag{13}$$

6. Homogene Maxwell-Gleichungen

$$\partial_\alpha \tilde{F}^{\alpha\beta} = 0 \quad \Leftrightarrow \quad \partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} + \partial_\gamma F_{\alpha\beta} = 0 \tag{14}$$

7. Inhomogene Maxwell-Gleichungen

$$\partial_\alpha F^{\alpha\beta} = -\mu_0 J^\beta \quad \Leftrightarrow \quad \Box A_\alpha - \partial_\alpha(\partial_\beta A^\beta) = -\mu_0 J_\alpha \tag{15}$$

8. Inhomogene Maxwell-Gleichungen in der Lorenz-Eichung

$$\partial_\alpha A^\alpha = 0 \quad \Rightarrow \quad \Box A_\alpha = -\mu_0 J_\alpha \tag{16}$$

9. Lorentz-Transformation des Feldstärketensors

$$\bar{x}^\alpha = L^\alpha_{\ \beta} x^\beta \quad \Rightarrow \quad \bar{F}_{\alpha\beta} = \Lambda_\alpha^{\ \gamma}\Lambda_\beta^{\ \delta} F_{\gamma\delta} \qquad \big(\,\Lambda = (L^T)^{-1}\,\big) \tag{17}$$

10. Invarianten (eichinvariante Lorentz-Skalare)

$$\begin{aligned} I_1 &= \tfrac{1}{4}F_{\alpha\beta}F^{\alpha\beta} = \tfrac{1}{2}(\vec{B}^2 - c^{-2}\vec{E}^2) \\ I_2 &= \tfrac{1}{4}F_{\alpha\beta}\tilde{F}^{\alpha\beta} = c^{-1}\vec{E}.\vec{B} \end{aligned} \tag{18}$$

11. Bewegungsgleichung für ein geladenes (Ladung q) massives (Masse m) Teilchen im elektro-magnetischen Feld (Lorentz-Kraft)

$$m\ddot{x}^\alpha = qF^\alpha_{\ \beta}\dot{x}^\beta \quad \Rightarrow \quad \frac{d}{dt}\vec{p} = q\left(\vec{E} + (\vec{v}\times\vec{B})\right) \qquad (\,\vec{p} = m\gamma(v)\vec{v}\,) \tag{19}$$

12. Wirkung für ein geladenes massives Teilchen im elektro-magnetischen Feld (minimale Kopplung)

$$S[x,A] = S_0[x] + S_I[x,A] = -mc^2\int d\tau + q\int d\tau\; A_\alpha \dot{x}^\alpha \tag{20}$$