4 Lorentz-Covariant Formulation of Maxwell Theory

4.1 Maxwell Equations (Review)

Maxwell Gleichungen in traditioneller Form

In the traditional (non-covariant, 3-vector) formulation, the Maxwell equations are the

1. Homogeneous Equations

HG: Homogene Gleichung

$$\begin{aligned} \vec{\nabla}.\vec{B} &= 0 \\ \vec{\nabla} \times \vec{E} + \partial_t \vec{B} &= 0 \end{aligned} \tag{4.1}$$

2. Inhomogeneous Equations

IG: Inhomogene Gleichung

$$\begin{aligned} \vec{\nabla}.\vec{E} &= \rho/\epsilon_0 \\ \vec{\nabla} \times \vec{B} - \frac{1}{c^2}\partial_t \vec{E} &= \mu_0 \vec{J} \end{aligned} \tag{4.2}$$

Ziemlich wirr, oder?

Here $\vec{E}$ and $\vec{B}$ are the electric and magnetic fields, and the sources of these fields are the electric charge density ρ and the current density $\vec{J}$. ϵ_0 and μ_0 are constants (whose names, let alone their values, I can never remember) which are related to the velocity of light by

$$\epsilon_0 \mu_0 = c^{-2} \quad . \tag{4.3}$$

The inhomogeneous equations imply the

3. Continuity Equation

KG: Kontinuitaetsgleichung

$$\partial_t \rho + \vec{\nabla}.\vec{J} = 0 \quad . \tag{4.4}$$

In the absence of sources, the homogeneous and inhomogeneous equations together imply the

4. Wave Equations for the Electric and Magnetic Fields

$$\rho = \vec{J} = 0 \quad \Rightarrow \quad \Box \vec{E} = 0 \quad , \quad \Box \vec{B} = 0 \quad . \tag{4.5}$$

In order to (locally) solve the homogeneous equations, and also for other purposes and reasons, it is useful to introduce the

Potentiale: werden eine wichtige Rolle spielen!

5. Electric Potential ϕ and Magnetic Potential $\vec{A}$

$$\begin{aligned} \vec{B} = \vec{\nabla} \times \vec{A} \quad &\Rightarrow \quad \vec{\nabla}.\vec{B} = 0 \\ \vec{E} = -\vec{\nabla}\phi - \partial_t \vec{A} \quad &\Rightarrow \quad \vec{\nabla} \times \vec{E} + \partial_t \vec{B} = 0 \end{aligned} \tag{4.6}$$

Introduction of these potentials gives rise to the

6. Gauge Transformations / Gauge Invariance

$$\phi \to \phi - \partial_t \Psi \quad , \quad \vec{A} \to \vec{A} + \vec{\nabla}\Psi \quad \Rightarrow \quad \vec{E} \to \vec{E} \quad , \quad \vec{B} \to \vec{B} \quad . \tag{4.7}$$

Finally, in terms of the potentials, the (remaining) inhomogeneous equations are the

7. Equations of Motion for the Potentials

$$\Box\vec{A} - \vec{\nabla}G = -\mu_0\vec{J}$$
$$\Box(-\phi/c) - \frac{1}{c}\partial_t G = \mu_0\rho c \tag{4.8}$$

IG für die Potentiale

with

$$G = \vec{\nabla}.\vec{A} + \frac{1}{c}\partial_t(\phi/c) \ . \tag{4.9}$$

This is all we will need.

Wo verstecken sich bitte die Lorentz-Tensoren?

4.2 Lorentz Invariance of the Maxwell Equations: Preliminary Remarks

At first sight, the presumed Lorentz invariance of the Maxwell equations, as presented above, and the possible Lorentz-tensorial structure of their building blocks are totally obscure. What we have are various 3-vectors (i.e. vectors under spatial rotations), such as $\vec{E}$ and $\vec{J}$, 3-vectorial differential operators like $\vec{\nabla}$, and 3-scalars like ϕ. So where do the Lorentz tensors hide?

The issue is particularly puzzling for the electric and magnetic fields $\vec{E}$ and $\vec{B}$: while the electromagnetic field of a charge at rest is purely electric, that of a charge moving with a constant velocity contains both electric and magnetic fields. This means that the decomposition of an electromagnetic field into electric and magnetic fields depends on the inertial system and that under Lorentz boosts electric and magnetic fields will "mix", i.e. transform into each other. How can one combine the 3 components of $\vec{E}$ and the 3 components of $\vec{B}$ into a Lorentz tensor?

However, looking a bit closer at these equations, one finds some suggestive and intriguing hints that these equations really want to be written in a much nicer four-dimensional Lorentz covariant way:

Alles sehr verwirrend, aber …

1. Our first clue comes from the continuity equation (4.4). We had already seen in section 2.10, that such an equation (2.179) is Lorentz invariant provided that ρ and $\vec{J}$ can be assembled into the components of a Lorentz 4-vector. This is indeed true in the case at hand and will be the starting point of our discussion below.

2. Our second clue will come from looking at the potentials: both the gauge transformations (4.7) and the wave equations (4.8) strongly suggest that ϕ and $\vec{A}$ should then also be collected into a Lorentz (co)vector.

3. Once we know how ϕ and $\vec{A}$ transform under Lorentz transformations, we can also determine how $\vec{E}$ and $\vec{B}$ transform under Lorentz transformations, i.e. how they are assembled into a Lorentz tensor (and, as we will see, the covariant formulation makes this particularly simple).

… schauen wir uns doch mal die KG an :

4.3 Electric 4-Current and Lorentz Invariance of the Continuity Equation

We recall from section 2.10 that, in terms of

$$J^a = (c\rho, \vec{J}) \ , \tag{4.10}$$

Hatten schon gesehen dass sich eine KG so schreiben laesst.
Aber ist J^a ein Lorentz-Vektor???

the continuity equation (4.4) can be written as (2.179)

$$\frac{\partial}{\partial t}\rho + \vec{\nabla}.\vec{j} = 0 \quad \Leftrightarrow \quad \partial_a J^a(x) = 0 \ . \tag{4.11}$$

and that this equation is Lorentz invariant if J^a is a Lorentz 4-vector.

In order to determine the transformation behaviour of the charge density ρ and current density $\vec{J}$ under Lorentz boost transformations, it is sufficient to consider charge densities moving at constant velocities. Our starting point and physical input will be the empirical fact that the (differential) charge dQ contained in a volume element dV is independent of its velocity. In the restframe of the charge distribution, say, one has

$$dQ = \rho_0 dV_0 \quad \text{and} \quad \vec{J}_0 = 0 \ . \tag{4.12}$$

Here ρ_0 is the rest charge density, and as such (tautologically) a scalar under Lorentz transformations, much like the rest mass of a particle. In an inertial system moving relative to the restframe at constant velocity v, one has a charge density ρ and a current density

$$\vec{J} = \rho\vec{v} \ . \tag{4.13}$$

Lorentz contraction

$$dV = \gamma(v)^{-1} dV_0 \tag{4.14}$$

and invariance of the charge,

$$dQ = \rho_0 dV_0 = \rho dV \tag{4.15}$$

imply

$$\rho = \gamma(v)\rho_0 \tag{4.16}$$

(this is intuitively obvious: smaller volume leads to larger charge density) and therefore

$$\vec{J} = \rho_0\gamma(v)\vec{v} \ . \tag{4.17}$$

Thus the components of J^a are

$$(J^a) = (c\rho, \vec{J}) = \rho_0(\gamma(v)c, \gamma(v)\vec{v}) \ . \tag{4.18}$$

Here we recognise the components (3.5) of the Lorentz vector 4-velocity u^a,

$$(u^a) = (\gamma(v)c, \gamma(v)\vec{v}) \ . \tag{4.19}$$

Since ρ_0 is a Lorentz scalar, we have established that

$$J^a = \rho_0 u^a \tag{4.20}$$

is indeed a Lorentz 4-vector, the electric 4-current (density) of Maxwell theory. In particular, therefore, the continuity equation is now manifestly Lorentz invariant.

Ja, J^a ist ein Lorentz-Vektor!!!

Remarks:

1. The argument given above for the 4-vector character of the current can also be applied to (discrete or continuous) distributions of relativistic particles: also in that case, the number density of particles ρ is such that $\rho/\gamma(v) = \rho_0$ is independent of the inertial system, and therefore
$$(J^a) = (c\rho, \rho\vec{v}) = \rho_0(u^a) \tag{4.21}$$
is a 4-vector.

2. For later convenience, we will henceforth also absorb the annoying constant μ_0 (cf. (4.8)) into the definition of the 4-current, i.e. we redefine
$$J^a = \mu_0\rho_0 u^a \quad , \tag{4.22}$$
with covariant components
$$(J_a) = (-\mu_0 c\rho, \mu_0\vec{J}) = (-\rho/(\epsilon_0 c), \mu_0\vec{J}) \quad . \tag{4.23}$$

Jetzt schauen wir uns die Gleichungen für die Potentiale an …

4.4 Inhomogeneous Maxwell Equations I: 4-Potential

Having identified ρ and $\vec{J}$ as components of a Lorentz 4-vector, looking back at the Maxwell equations (4.8) and gauge transformations (4.7) strongly suggests to also combine the electric and magnetic potentials ϕ and $\vec{A}$ into a 4-component object.

Indeed, let us set
$$(A_a) = (-\phi/c, \vec{A}) \quad . \tag{4.24}$$

Then the first obervation is that the gauge transformations (4.7) can uniformly and elegantly be written as
$$\phi \to \phi - \partial_t\Psi \quad , \quad \vec{A} \to \vec{A} + \vec{\nabla}\Psi \quad \Leftrightarrow \quad A_a \to A_a + \partial_a\Psi \tag{4.25}$$
for an arbitrary function $\Psi = \Psi(x)$ on Minkowski space. We also see that the function G introduced in (4.9) can simply be written as
$$G = \vec{\nabla}.\vec{A} + \frac{1}{c}\partial_t(\phi/c) = \partial_a A^a \tag{4.26}$$
(note that $(A^a) = (+\phi/c, \vec{A})$). With this, and the definition of the current J_a (including the factor of μ_0) we can write the equations of motion for the potentials (4.8) collectively and simply as
$$\Box A_a - \partial_a(\partial_b A^b) = -J_a \quad . \tag{4.27}$$

Now, since $\Box$ is a Lorentz scalar, and ∂_a and J_a are Lorentz covectors, this equation will be Lorentz invariant if and only if A_a transforms as a Lorentz covector (and thus $\partial_b A^b$ is a Lorentz scalar).

We have thus, with very little effort, managed to write the inhomogeneous Maxwell equations in a manifestly Lorentz invariant form.

Die IG sind Lorentz-invariant!!! ;-)

Remarks:

1. The gauge transformation behaviour (4.25)

$$A_a \to A_a + \partial_a \Psi \tag{4.28}$$

shows that the 4-potential should naturally be thought of as a covector A_a rather than as a vector A^a.

2. The result (4.27) is manifestly Lorentz invariant. It is also gauge invariant, as it has to be: under $A_a \to A_a + \partial_a \Psi$ one has

$$\Box A_a - \partial_a(\partial_b A^b) \to \Box A_a + \Box \partial_a \Psi - \partial_a(\partial_b A^b) - \partial_a(\partial_b \partial^b \Psi) = \Box A_a - \partial_a(\partial_b A^b) \tag{4.29}$$

(because partial derivatives commute). However, gauge invariance is not yet manifest, and we will rectify this in the next section (after having introduced the Maxwell field strength tensor). This field strength tensor will then also allow us to immediately read off the transformation behaviour of the electric and magnetic fields under Lorentz transformations.

3. The term $G = \partial_b A^b$ by itself is evidently not gauge invariant. A convenient gauge condition is the so-called Lorenz gauge (without the "t", named after Ludwig Lorenz, not Hendrik Lorentz)

$$G = \partial_a A^a = 0 \ . \tag{4.30}$$

Not only do the Maxwell equations decouple in this gauge,

$$G = 0 \quad \Rightarrow \quad \Box A_a = -J_a \tag{4.31}$$

(so that the general solution can immediately be written down in terms of Greens functions for the wave operator $\Box$). This gauge condition is also the (essentially unique) gauge condition on A_a that perserves Lorentz invariance (other common gauge conditions like the Coulomb gauge, $\vec{\nabla}.\vec{A} = 0$, or axial gauges like $A_0 = 0$, are evidently not Lorentz invariant).

Aber es geht noch besser: manifest Lorentz- und Eich-invariant!

4.5 Inhomogeneous Maxwell Equations II: Maxwell Field Strength Tensor

We now want to find out how to express the gauge invariant fields $\vec{E}$ and $\vec{B}$ in a Lorentz tensorial way. To that end we start with the observation that

$$\vec{E} = -\vec{\nabla}\phi - \partial_t \vec{A} \quad , \quad \vec{B} = \vec{\nabla} \times \vec{A} \tag{4.32}$$

are precisely those linear combinations of the first partial derivatives of the potentials ϕ and $\vec{A}$ that are gauge invariant. Thus, as our first step we determine how the first derivatives $\partial_a A_b$ of A_b transform under gauge transformations:

$$A_b \to A_b + \partial_b \Psi \quad \Rightarrow \quad \partial_a A_b \to \partial_a A_b + \partial_a \partial_b \Psi \ . \tag{4.33}$$

We see that in general the partial derivatives of A_b are not gauge invariant, as expected. But the offending term

$$\partial_a \partial_b \Psi = \partial_b \partial_a \Psi \tag{4.34}$$

E und B sind eichinvariante lineare Kombinationen von Ableitungen der Potentiale - also versuchen wir doch etwas ähnliches mit dem 4er-Potential A_b

has the one characteristic property that it is symmetric (because partial derivatives commute ...). Therefore, we can eliminate it by taking the anti-symmetrised derivative of A_b,

$$A_b \to A_b + \partial_b \Psi \quad \Rightarrow \quad \partial_a A_b - \partial_b A_a \to \partial_a A_b - \partial_b A_a \ . \tag{4.35}$$

These are now precisely the gauge invariant linear combinations of the first derivatives of the potentials, and thus they must be expressible in terms of $\vec{E}$ and $\vec{B}$ (and we will verify this shortly). In any case, this motivates us to define and introduce the Maxwell field strength tensor

Maxwell Feldstärke Tensor F:

$$F_{ab} = \partial_a A_b - \partial_b A_a \ . \tag{4.36}$$

F ist eichinvariant!!!

In addition to gauge invariance, F_{ab} has the following two important properties:

- F_{ab} is anti-symmetric, $F_{ab} = -F_{ba}$. Thus it has 6 independent components, precisely the right number to accommodate $\vec{E}$ and $\vec{B}$: this is how two 3-vectors can combine into a Lorentz tensor!
- F_{ab} is a Lorentz (0,2)-tensor, i.e. under Lorentz transformations $\bar{x}^a = L^a_{\ b} x^b$ it transforms as

$$\bar{F}_{ab}(\bar{x}) = \Lambda_a^{\ c} \Lambda_b^{\ d} F_{cd}(x) \ . \tag{4.37}$$

Combining these two facts, we see that once we have determined the relation between the components of F_{ab} and those of $\vec{E}$ and $\vec{B}$, the Lorentz transformation of $\vec{E}$ and $\vec{B}$ is determined (and reduces to simple matrix multiplication).

Thus let us now determine the relation between F_{ab} and $\vec{E}, \vec{B}$. To that end, we first write the defining relations (4.32) in components as

$$E_i = -\partial_i \phi - \partial_t A_i \quad , \quad B_i = \epsilon_{ijk} \partial_j A_k \ \Leftrightarrow \ \partial_i A_j - \partial_j A_i = \epsilon_{ijk} B_k \tag{4.38}$$

Komponenten von F müssen sich also durch E und B ausdrücken lassen

(I am deliberately not careful with the positioning of the spatial indices here, summation over repeated indices is still understood). Now we turn to the components of F_{ab} in this inertial system. Since F_{ab} is anti-symmetric, with

$$(A_a) = (-\phi/c, \vec{A}) \tag{4.39}$$

the independent components are

$$\begin{aligned} F_{0i} &= \partial_0 A_i - \partial_i A_0 = -E_i/c = -F_{i0} \\ F_{ij} &= \partial_i A_j - \partial_j A_i = \epsilon_{ijk} B_k \ . \end{aligned} \tag{4.40}$$

Thus, as expected, F_{ab} can be expressed entirely and easily in terms of the electric and magnetic fields. In matrix form, one can also write this as

$$(F_{ab}) = \begin{pmatrix} 0 & -E_1/c & -E_2/c & -E_3/c \\ +E_1/c & 0 & +B_3 & -B_2 \\ +E_2/c & -B_3 & 0 & +B_1 \\ +E_3/c & +B_2 & -B_1 & 0 \end{pmatrix} \tag{4.41}$$

It will also be useful to know the contravariant components

$$F^{ab} = \eta^{ac}\eta^{bd}F_{cd} \quad . \tag{4.42}$$

For these one has

$$F^{0i} = -F_{0i} \quad , \quad F^{ij} = F_{ij} \quad , \tag{4.43}$$

and thus

$$(F^{ab}) = \begin{pmatrix} 0 & +E_1/c & +E_2/c & +E_3/c \\ -E_1/c & 0 & +B_3 & -B_2 \\ -E_2/c & -B_3 & 0 & +B_1 \\ -E_3/c & +B_2 & -B_1 & 0 \end{pmatrix} \tag{4.44}$$

Next we want to write the inhomogeneous Maxwell equations (4.27)

$$\Box A_b - \partial_b(\partial_a A^a) = -J_b \tag{4.45}$$

in terms of F_{ab}. Since F_{ab} is constructed from the first derivatives of A_a, we need to look at first derivatives of F_{ab}, and the result should be a covector. There is really only one possibility, namely $\partial^a F_{ab}$. Working this out, one finds that on the nose

Jetzt IG mit F schreiben:

$$\partial^a F_{ab} = \partial^a\partial_a A_b - \partial^a\partial_b A_a = \Box A_b - \partial_b(\partial_a A^a) \quad . \tag{4.46}$$

Thus we can write the Maxwell equations in the simple and beautiful form

$$\partial^a F_{ab} = -J_b \quad \Leftrightarrow \quad \partial_a F^{ab} = -J^b \quad . \tag{4.47}$$

This is the sought-for manifestly Lorentz and gauge invariant formulation of the Maxwell equations.

So einfach und zwingend logisch (was könnte es sonst sein) kann man also die IG schreiben! ;-)

<u>Remarks:</u>

1. Using the explicit expression for the components of F^{ab} given above, it is straightforward to also verify directly that these equations are equivalent to the inhomogeneous Maxwell equations (4.2),

$$\partial_a F^{ab} = -J^a \quad \Leftrightarrow \quad \vec{\nabla}.\vec{E} = \rho/\epsilon_0 \quad , \quad \vec{\nabla}\times\vec{B} - \frac{1}{c^2}\partial_t\vec{E} = \mu_0\vec{J} \quad . \tag{4.48}$$

   For example,

$$\partial_a F^{a0} = \partial_i F^{i0} = -\partial_i E_i/c = -\rho/(\epsilon_0 c) = -\mu_0\rho c = -J^0 \tag{4.49}$$

   and likewise for the spatial components $\partial_a F^{aj}$.

2. The continuity equation $\partial_a J^a = 0$ follows trivially from (4.47):

Beweis der KG ist eine 1-Zeilen Rechnung!

$$\partial_b J^b = -\partial_b\partial_a F^{ab} = 0 \tag{4.50}$$

   beacuse $\partial_b\partial_a$ is symmetric (partial derivatives commute ...) and F^{ab} is anti-symmetric.

(und man braucht keine seltsamen Vektoranalysis Identitäten dafür, wie DivRot=0 oder RotGrad = 0 sondern nur die Regel: partielle Ableitungen vertauschen ...)

HG: Nächste Woche …

4.6 Homogeneous Maxwell Equations I: Bianchi Identities

Looking back at the Maxwell equations recalled in section 4.1, we see that the only equations that we have not yet cast into manifestly Lorentz-invariant form are the homogeneous equations (4.1). One way to approach the question how to do go about this is to note that these equations are identically satisfied once one has introduced the potentials. In the present context, we are thus asking the question what differential equations are identically satisifed by an F_{ab} of the form $F_{ab} = \partial_a A_b - \partial_b A_a$.

- As a warm-up exercise (with one index less), let us consider the question what sort of differential equations are identically satisfied by a covector $F_a = \partial_a A$. In that case the well-known answer is that its anti-symmetrised derivative is zero

$$F_a = \partial_a A \quad \Rightarrow \quad \partial_a F_b - \partial_b F_a = \partial_a \partial_b A - \partial_b \partial_a A = 0 \tag{4.51}$$

  (partial derivatives commute ...).

- The same strategy works for $F_{ab} = \partial_a A_b - \partial_b A_a$: since partial derivatives commute, the totally anti-symmetrised derivative of F_{ab} will be identically zero,

$$F_{ab} = \partial_a A_b - \partial_b A_a \quad \Rightarrow \quad \partial_a F_{bc} - \partial_b F_{ac} + \text{ 4 more terms } = 0 \ . \tag{4.52}$$

  In general, such identities, resulting from anti-symmetrisation of differential operators, are referred to as *Bianchi Identities*.

  Using the results and notation of section 2.8, in particular the identity (2.143),

$$T_{abc} = T_{a[bc]} \quad \Rightarrow \quad T_{[abc]} = \tfrac{1}{3}(T_{abc} + T_{cab} + T_{bca}) \ , \tag{4.53}$$

  we can write this as

$$F_{ab} = \partial_a A_b - \partial_b A_a \quad \Rightarrow \quad \partial_{[a} F_{bc]} = 0 \quad \Leftrightarrow \quad \partial_a F_{bc} + \partial_b F_{ca} + \partial_c F_{ab} = 0 \ . \tag{4.54}$$

  The fact that the equation on the left implies the equation on the right is also easily verified directly.

While these equations, with their 3 indices, look somewhat intransparent (and of course we will improve that below!), already now we can verify that these are precisely 4 independent equations, and that, with F_{ab} expressed in terms of $\vec{E}$ and $\vec{B}$, they reproduce precisely the homogeneous Maxwell equations,

$$\partial_a F_{bc} + \partial_b F_{ca} + \partial_c F_{ab} = 0 \quad \Leftrightarrow \quad \vec{\nabla} \times \vec{E} + \partial_t \vec{B} = 0 \ , \ \vec{\nabla}.\vec{B} = 0 \ . \tag{4.55}$$

We need to consider 3 different cases:

1. two indices are equal

   We first observe that the equations on the left-hand side are empty (trivially satisfied for any anti-symmetric F_{ab}) if any 2 indices are equal (since the left-hand side is totally anti-symmetric, this could hardly be otherwise). Indeed, if $a = b$, say, then we have

$$\partial_a F_{ac} + \partial_a F_{ca} + \partial_c F_{aa} = \partial_a F_{ac} - \partial_a F_{ac} + 0 = 0 \tag{4.56}$$

   identically, just by anti-symmetry of F_{ab}. Thus all 3 indices have to be different.

2. all indices are spatial, e.g. $(a = 1, b = 2, c = 3)$

   In this case one has
$$\partial_1 F_{23} + \partial_2 F_{31} + \partial_3 F_{12} = \vec{\nabla}.\vec{B} \ . \tag{4.57}$$

3. one index is temporal and the others are spatial, e.g. $(a = 0, b = 1, c = 2)$ (or essentially, up to signs and permutations, two more possibilities)

   In this case one has
$$\partial_0 F_{12} + \partial_1 F_{20} + \partial_2 F_{01} = c^{-1}(\partial_t \vec{B} + \nabla \times \vec{E})_3 \tag{4.58}$$
   (and likewise for the remaining components).

This establishes (4.55).

Thus we can neatly summarise basically all of Maxwell theory by
$$\text{Maxwell Equations:} \quad \begin{cases} \partial_a F^{ab} = -J^b \\ \partial_{[a} F_{bc]} = 0 \end{cases} \tag{4.59}$$

A famous consequence of the Maxwell equations is that, in source-free regions of space(-time) the electric and magnetic fields propagate as waves with velocity c,
$$\rho = \vec{J} = 0 \quad \Rightarrow \quad \Box \vec{E} = \Box \vec{B} = 0 \ . \tag{4.60}$$

The usual non-covariant 3-vector calculus derivation of this is somewhat roundabout, and requires the full set of eight (homogeneous and inhomogeneous) Maxwell equations and judicious use of various 3-vector calculus identities. Here is a 1-line proof of the statement
$$\partial^a F_{ab} = -J_b = 0 \quad \Rightarrow \quad \Box F_{ab} = 0 \tag{4.61}$$
in our formulation:
$$0 = \partial^c(\partial_a F_{bc} + \partial_b F_{ca} + \partial_c F_{ab}) = \partial_a \partial^c F_{bc} + \partial_b \partial^c F_{ca} + \Box F_{ab} = \Box F_{ab} \ . \tag{4.62}$$

When the 4-current is not equal to zero, one has instead
$$\Box F_{ab} = \partial_b J_a - \partial_a J_b \ . \tag{4.63}$$

4.7 Homogeneous Maxwell Equations II: Dual Field Strength Tensor

While the form of the homogeneous Maxwell equation given in (4.59) is nicely manifestly Lorentz- and gauge invariant, there is a different way of writing it which makes it more manifest that these are indeed only precisely four equations, and which brings out a nice analogy between the homogeneous and inhomgeneous equations.

Recall that already in ordinary 3-vector calculus, frequently, instead of anti-symmetrising explicitly, it is much more convenient to let the ϵ- (or Levi-Civita) symbol ϵ_{ijk} do the job, as in
$$\partial_j A_k - \partial_k A_j \to \epsilon_{ijk} \partial_j A_k \equiv B_i \ . \tag{4.64}$$

In particular, then the identity $\vec{\nabla}.\vec{B} = 0$ becomes manifest because (once again ...) partial derivatives commute,

$$\partial_i B_i = \epsilon_{ijk}\partial_i\partial_j A_k = 0 \ . \tag{4.65}$$

In this 3-dimensional case, all the components of ϵ_{ijk} are determined by total anti-symmetry and the choice (of orientation) $\epsilon_{123} = 1$,

$$\epsilon_{ijk} = \epsilon_{[ijk]} \quad , \quad \epsilon_{123} = 1 \ . \tag{4.66}$$

In our 4-dimensional case, we can analogously introduce a totally anti-symmetric spacetime ϵ-symbol ϵ_{abcd} by

$$\epsilon_{abcd} = \epsilon_{[abcd]} \quad , \quad \epsilon_{0123} = +1 \ . \tag{4.67}$$

To be compatible with our conventions for raising and lowering indices, we also define ϵ^{abcd} by

$$\epsilon^{abcd} = \epsilon^{[abcd]} \quad , \quad \epsilon^{0123} = -1 \ . \tag{4.68}$$

Then, letting ϵ^{abcd} taking care of the total anti-symmetrisation, we can write the homogeneous Maxwell equations as

$$\partial_{[a}F_{cd]} = 0 \quad \Leftrightarrow \quad \epsilon^{abcd}\partial_a F_{cd} = \partial_a(\epsilon^{abcd}F_{cd}) = 0 \ . \tag{4.69}$$

We are thus led to introduce the *dual Maxwell field strength tensor* $\tilde{F}^{ab}$ by (the factor of 1/2 is a convenient convention)

$$\tilde{F}^{ab} = \tfrac{1}{2}\epsilon^{abcd}F_{cd} \ . \tag{4.70}$$

Then we have

$$\partial_{[a}F_{cd]} = 0 \quad \Leftrightarrow \quad \partial_a\tilde{F}^{ab} = 0 \ , \tag{4.71}$$

and it is now manifest that these are indeed precisely 4 equations.

Thus we can write the full set of Maxwell equations as

$$\text{Maxwell Equations:} \quad \begin{cases} \partial_a F^{ab} = -J^b \\ \partial_a \tilde{F}^{ab} = 0 \end{cases} \tag{4.72}$$

<u>Remarks:</u>

1. The dual field strength tensor $\tilde{F}^{ab}$ is, i.e. transforms as, a tensor under rotations and boosts (the transformations that we usually call Lorentz transformations), but because a choice of orientation is involved in the definition of ϵ_{abcd}, it transforms additionally with a sign $\det(L) = \pm 1$ under general Lorentz transformations. This is just like in 3-dimensional vector calculus, where the vector product, defined with the help of ϵ_{ijk} defines not a vector but what is known as a pseudo-vector (sensitive to the orientation: right-hand versus left-hand rule). For the time being, however, since we are not interested in space or time reflections, we can ignore this subtlety.

2. Explicitly, the components of $\tilde{F}^{ab}$ are related to those of F_{ab} e.g. by

$$\begin{aligned} \tilde{F}^{01} &= \tfrac{1}{2}\epsilon^{01cd}F_{cd} = \tfrac{1}{2}(\epsilon^{0123}F_{23} + \epsilon^{0132}F_{32}) = \epsilon^{0123}F_{23} = -F_{23} \\ \tilde{F}^{23} &= \tfrac{1}{2}\epsilon^{23cd}F_{cd} = \epsilon^{2301}F_{01} = \epsilon^{0123}F_{01} = -F_{01} \end{aligned} \tag{4.73}$$

etc. In terms of $\vec{E}$ and $\vec{B}$ this means

$$\tilde{F}^{01} = -B_1 \quad , \quad \tilde{F}^{23} = E_1/c \tag{4.74}$$

etc., so that we can write $\tilde{F}^{ab}$ in matrix form as

$$(\tilde{F}^{ab}) = \begin{pmatrix} 0 & -B_1 & -B_2 & -B_3 \\ +B_1 & 0 & +E_3/c & -E_2/c \\ +B_2 & -E_3/c & 0 & +E_1/c \\ +B_3 & +E_2/c & -E_1/c & 0 \end{pmatrix} \tag{4.75}$$

3. One can now also verify directly that

$$\partial_a \tilde{F}^{ab} = 0 \quad \Leftrightarrow \quad \vec{\nabla} \times \vec{E} + \partial_t \vec{B} = 0 \ , \ \vec{\nabla}.\vec{B} = 0 \ . \tag{4.76}$$

E.g.

$$\partial_a \tilde{F}^{a0} = \partial_i \tilde{F}^{i0} = \partial_i B_i = \vec{\nabla}.\vec{B} \tag{4.77}$$

(and likewise for the other components).

4. Comparison with (F^{ab}) (4.44),

$$(F^{ab}) = \begin{pmatrix} 0 & +E_1/c & +E_2/c & +E_3/c \\ -E_1/c & 0 & +B_3 & -B_2 \\ -E_2/c & -B_3 & 0 & +B_1 \\ -E_3/c & +B_2 & -B_1 & 0 \end{pmatrix} \tag{4.78}$$

shows that $\tilde{F}^{ab}$ is obtained from F^{ab} by sending

$$F^{ab} \to \tilde{F}^{ab} \quad \Leftrightarrow \quad \vec{E}/c \to -\vec{B} \quad \text{and} \quad \vec{B} \to \vec{E}/c \ . \tag{4.79}$$

Thus this exchanges the electric and magnetic fields.

5. In fact, this transformation is known as the electric-magnetic duality transformation of Maxwell theory. You may have noticed before the curious fact that the Maxwell equations (without electric sources) are invariant under this transformation, i.e. the homogeneous equations get mapped to the inhomogeneous equations (without sources) and vice versa: it is obvious that the transformation exchanges

$$\vec{\nabla}.\vec{E} = 0 \quad \leftrightarrow \quad \vec{\nabla}.\vec{B} = 0 \ , \tag{4.80}$$

but it is also true that it exchanges the remaining equations, since

$$\vec{\nabla} \times \vec{B} - \frac{1}{c}\partial_t(\vec{E}/c) \quad \leftrightarrow \quad (\partial_t \vec{B} + \vec{\nabla} \times \vec{E})/c \ . \tag{4.81}$$

6. In the present formulation, this duality symmetry of the vacuum equations could not be more obvious. In the absence of electric sources, the Maxwell equations read

$$J^a = 0 \quad \Rightarrow \quad \partial_a F^{ab} = 0 \quad , \quad \partial_a \tilde{F}^{ab} = 0 \ , \tag{4.82}$$

which are manifestly invariant under the exchange $F^{ab} \leftrightarrow \tilde{F}^{ab}$. Unfortunately, in the presence of sources, this nice and intriguing duality symmetry is broken by the (unexplained) absence of magnetic monopole charges and currents in the real world.

4.8 Maxwell Theory and Lorentz Transformations I: Lorentz Scalars

Now that we know how the Maxwell field strength tensor F_{ab} transforms under Lorentz transformations, namely as a (0,2)-tensor, and how the components of F_{ab} are related to those of $\vec{E}$ and $\vec{B}$, we can now easily determine the transformation behaviour of $\vec{E}$ and $\vec{B}$ under Lorentz transformations, and we will come back to this below.

However, as always, it is useful to first think about and look for and at Lorentz scalars, i.e. objects that are actually invariant under Lorentz transformations. With the building blocks A_a and F_{ab} at our disposal, one Lorentz scalar that we could construct is

$$A_a A^a = \eta^{ab} A_a A_b \ , \tag{4.83}$$

but while this is a Lorentz scalar, it is not invariant under gauge transformations, and therefore of no interest to us. If we require gauge invariance in addition to Lorentz invariance, then we need to work with F_{ab}. The most obvious strategy to construct a scalar out of a $(0,2)$-tensor is (cf. the discussion in section 2.8) to take its η-trace, but beacuse F_{ab} is anti-symmetric, this will vanish,

$$F^a_{\ a} \equiv \eta^{ab} F_{ab} = 0 \ . \tag{4.84}$$

Thus there are no gauge invariant Lorentz scalars that are linear functions of $\vec{E}$ and $\vec{B}$. However, it is easy to construct a scalar that is quadratic in F_{ab}, namely

$$I_1 = \tfrac{1}{4} F_{ab} F^{ab} = \tfrac{1}{4} \eta^{ac} \eta^{bd} F_{ab} F_{cd} \tag{4.85}$$

(the factor of 1/4 is just a convention). Expressed in terms of $\vec{E}$ and $\vec{B}$, this is

$$I_1 = \tfrac{1}{4}(F_{0k}F^{0k} + F_{k0}F^{k0} + F_{ik}F^{ik}) = \tfrac{1}{2}(\vec{B}^2 - \vec{E}^2/c^2) \ . \tag{4.86}$$

The fact that this is a Lorentz scalar has some immediate consequences. Namely, if there is one inertial system in which $I_1 > 0$ (or $I_1 = 0$ or $I_1 < 0$), then in all inertial systems $I_1 > 0$ (or $I_1 = 0$ or $I_1 < 0$).

For example, consider the electromagnetic field of a charge at rest in some inertial system. In that inertial system, $\vec{E} \neq 0$ but $\vec{B} = 0$. In particular, therefore, I_1 is negative, $I_1 < 0$. In some other inertial system, it is clear that there will be both an electric and a magnetic field, but the additional information that the invariant I_1 provides us with, without any further calculation, is that the magnetic field cannot exceed the electric field in magnitude,

$$I_1 = \bar{I}_1 < 0 \quad \Rightarrow \quad |\overline{\vec{B}}| < |\overline{\vec{E}}|/c \ . \tag{4.87}$$

There is another invariant that we can construct, namely

$$I_2 = \tfrac{1}{4} F_{ab}\tilde{F}^{ab} \quad . \tag{4.88}$$

This is a scalar under rotations and boosts (but, like $\tilde{F}^{ab}$, transforms with the sign $\det L$ under general more general Lorentz transformations). Expressed in terms of $\vec{E}$ and $\vec{B}$, this is

$$I_2 = \vec{B}.\vec{E}/c \quad . \tag{4.89}$$

In particular this implies that if e.g. $\vec{B} = 0$ in some inertial system, then in any inertial system the electric field will be orthogonal to the magnetic field. As regards the above example of a moving charge, this provides us with the additional information that the magnetic field of a moving charge will be orthogonal to its electric field.

One property of I_2 that we will come back to later in our discussion of an action principle for Maxwell theory is the fact that when $F_{ab} = \partial_a A_b - \partial_b A_a$, the invariant I_2 can (unlike I_1) be written as a total derivative. Indeed, writing

$$F_{ab}\tilde{F}^{ab} = \tfrac{1}{2}\epsilon^{abcd}F_{ab}F_{cd} = \epsilon^{abcd}F_{ab}\partial_c A_d \quad , \tag{4.90}$$

we see that this can be written as

$$F_{ab}\tilde{F}^{ab} = \partial_c(\epsilon^{abcd}F_{ab}A_d) - \epsilon^{abcd}(\partial_c F_{ab})A_d = \partial_c(\epsilon^{abcd}F_{ab}A_d) \quad , \tag{4.91}$$

where in the last step we used the Bianchi identity satisfied by F_{ab}.

Are there any further (independent) invariants we can construct? The answer is no (and one can prove this using group theory, but we shall not do this here). Here are some examples to illustrate this claim:

1. The most obvious candidate for another invariant is perhaps the square of the dual field strength tensor $\tilde{F}^{ab}$, but it is easy to see that

$$\tilde{I}_1 \equiv \tfrac{1}{4}\tilde{F}_{ab}\tilde{F}^{ab} = -\tfrac{1}{4}F_{ab}F^{ab} = -I_1 \quad . \tag{4.92}$$

2. Any scalar constructed from an odd number of F_{ab} and/or $\tilde{F}^{ab}$ is automatically zero (because it can be regarded as the trace of an odd number of anti-symmetric matrices, which is zero). For example,

$$I_3 = F^a_{\ b}F^b_{\ c}F^c_{\ a} = 0 \quad . \tag{4.93}$$

3. Scalars constructed from an even nunmber of F_{ab} and/or $\tilde{F}^{ab}$ can be expressed in terms of polynomials of I_1 and I_2. For example, for

$$I_4 = F^{ab}F_{bc}F^{cd}F_{da} \tag{4.94}$$

one finds, after an uninspiring but straightforward calculation, something like

$$I_4 = 8(I_1)^2 + 4(I_2)^2 \quad . \tag{4.95}$$

4.9 Maxwell Theory and Lorentz Transformations II: Transformation of $\vec{E}, \vec{B}$

Finally, we turn to the simple (and purely algebraic) task of determining the transformation behaviour of $\vec{E}$ and $\vec{B}$ under Lorentz transformations. In general we already know that F_{ab} transforms like a (0,2) tensor field, i.e.

$$\bar{x}^a = L^a{}_b x^b \quad \Rightarrow \quad \bar{F}_{ab}(\bar{x}) = \Lambda_a{}^c \Lambda_b{}^d F_{cd}(x) \ . \tag{4.96}$$

As they stand, the above equations express the new fields at $\bar{x}$ in terms of the old fields at x. In order to express the new fields as functions of $\bar{x}$, as one would presumably like, all one needs to do is to write the x^a as

$$x^a = (L^{-1})^a{}_b \bar{x}^b \ , \tag{4.97}$$

so that

$$\bar{F}_{ab}(\bar{x}) = \Lambda_a{}^c \Lambda_b{}^d F_{cd}(L^{-1}\bar{x}) \ . \tag{4.98}$$

Under spatial rotations, $\vec{E}$ and $\vec{B}$ transform in the familiar was as 3-vectors. Thus we only need to look at Lorentz boosts, and without loss of generality we consider a boost in the x^1-direction, which has the form (cf. section 2.4)

$$(L^a{}_b) = \begin{pmatrix} \cosh\alpha & -\sinh\alpha & 0 & 0 \\ -\sinh\alpha & \cosh\alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{4.99}$$

with

$$\cosh\alpha(v) = \gamma(v) \quad , \quad \sinh\alpha(v) = \beta(v)\gamma(v) \ . \tag{4.100}$$

Therefore, $\Lambda = (L^T)^{-1}$ has the form

$$\left(\Lambda_a{}^b\right) = \begin{pmatrix} \cosh\alpha & \sinh\alpha & 0 & 0 \\ \sinh\alpha & \cosh\alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{4.101}$$

It follows that e.g. (suppressing the argument x or $\bar{x}$ for simplicity and for the time being)

$$\begin{aligned} \bar{F}_{01} &= \Lambda_0{}^c \Lambda_1{}^d F_{cd} = (\Lambda_0{}^0 \Lambda_1{}^1 - \Lambda_0{}^1 \Lambda_1{}^0) F_{01} = F_{01} \\ \bar{F}_{02} &= \Lambda_0{}^c \Lambda_2{}^d F_{cd} = \Lambda_0{}^c F_{c2} = \cosh\alpha F_{02} + \sinh\alpha F_{12} \\ \bar{F}_{12} &= \Lambda_1{}^c \Lambda_2{}^d F_{cd} = \Lambda_1{}^c F_{c2} = \sinh\alpha F_{02} + \cosh\alpha F_{12} \end{aligned} \tag{4.102}$$

etc. In terms of the components of the electric and magnetic fields one thus has

$$\begin{aligned} \bar{E}_1 &= E_1 \quad , \quad \bar{E}_2 = \gamma(E_2 - \beta c B_3) \quad , \quad \bar{E}_3 = \gamma(E_3 + \beta c B_2) \\ \bar{B}_1 &= B_1 \quad , \quad \bar{B}_2 = \gamma(B_2 + \beta E_3/c) \ , \quad \bar{B}_3 = \gamma(B_3 - \beta E_2/c) \end{aligned} \tag{4.103}$$

We see that the "longitudinal" components of the fields are not changed by a boost, while the transverse components are deformed.

If we want to reinstate the dependence of the fields on the coordinates, then we proceed as in (4.98) above. In the case at hand, since L is symmetric, the components of L^{-1} are just those of Λ.

When originally there is just an electric field, these equations simplify to

$$\vec{B} = 0 \quad \Rightarrow \quad \overline{\vec{E}} = (E_1, \gamma E_2, \gamma E_3) \quad , \quad \overline{\vec{B}} = (0, \beta\gamma E_3/c, -\beta\gamma E_2/c) \tag{4.104}$$

and one can explicitly check the assertions regarding the invariants I_1 and I_2 made in the previous section, e.g. the fact that the new magnetic field is orthogonal to the new electric field.

4.10 Example: The Field of a Moving Charge (Outline)

One can now use these methods to solve in a very simple way some standard problems of electrodynamics, e.g. to determine the electromagnetic field created by a charge or current moving with constant velocity. To that end,

- one first solves the problem in the rest frame of the charge or current (so in this case this is the simple electrostatics problem of determining the electric field of a static charge or a charged wire)
- and one then applies a Lorentz transformation to this solution to obtain the electromagnetic field of the moving charge or electric current.

The only thing one has to pay attention to is, as mentioned above, the correct assignment of the coordinates to the fields.

Concretely, assume that a point particle with charge q is at rest at the origin of the inertial system with coordinates $x^a = (ct, \vec{x})$. Then it has a purely electric and time-independent field given by the solution to $\vec{\nabla}.\vec{E} = \rho/\epsilon_0$, namely

$$\vec{E}(\vec{x}) = Q\frac{\vec{x}}{|\vec{x}|^3} \quad , \tag{4.105}$$

where I have introduced the abbreviation

$$Q = \frac{q}{4\pi\epsilon_0} \quad . \tag{4.106}$$

It follows from the above formulae that in the inertial system with coordinates $\bar{x}^a$ (with respect to which the charge moves with constant velocity $-v$ in the $\bar{x}^1$-direction, apologies for the minus sign ...), the electric field is given by

$$\begin{aligned} \bar{E}_1(\bar{x}) &= E_1(x) = Q\frac{x^1}{|\vec{x}|^3} \\ \bar{E}_2(\bar{x}) &= \gamma E_2(x) = \gamma Q\frac{x^2}{|\vec{x}|^3} \\ \bar{E}_3(\bar{x}) &= \gamma E_3(x) = \gamma Q\frac{x^3}{|\vec{x}|^3} \quad . \end{aligned} \tag{4.107}$$

Thus all that is left to do is to express the spatial coordinates x^i on the right-hand side in terms of the spacetime coordinates $\bar{x}^a$ via the inverse Lorentz transformation. One can of course do this in general but, in order to simplify the subsequent formulae, let us choose an observer at rest in the new inertial system at a point P with spatial coordinates

$$\bar{x}^i_P = (0, \bar{x}^2 = b, 0) \ . \tag{4.108}$$

In terms of the coordinates x^i, this observer has the coordinates

$$x^i_P = (\gamma(v)\beta(v)\bar{x}^0, b, 0) = (\gamma(v)v\bar{t}, b, 0) \ . \tag{4.109}$$

In particular,

$$|\vec{x}_P| = (\gamma^2 v^2 \bar{t}^2 + b^2)^{1/2} \ . \tag{4.110}$$

Putting everything together, we find that in the inertial system in which the observer is at rest (and the charge moves with constant velocity), the observer sees a time-dependent electric field given by

$$\begin{aligned} \bar{E}_1(\bar{x}^i_P, \bar{t}) &= Q\frac{\gamma(v)v\bar{t}}{(\gamma^2 v^2 \bar{t}^2 + b^2)^{3/2}} \\ \bar{E}_2(\bar{x}^i_P, \bar{t}) &= Q\frac{\gamma(v)b}{(\gamma^2 v^2 \bar{t}^2 + b^2)^{3/2}} \\ \bar{E}_3(\bar{x}^i_P, \bar{t}) &= 0 \ . \end{aligned} \tag{4.111}$$

We see that the transverse component $\bar{E}_2$ reaches its maxmimum at the time $\bar{t} = 0$ (the time when the distance between the charge and the observer takes on its minimal value), with

$$\bar{E}_2(\bar{x}^i_P, \bar{t} = 0) = \frac{Q\gamma(v)}{b^2} \tag{4.112}$$

proportional to $\gamma(v)$, and hence large for a rapidly moving charge. The longitudinal component $\bar{E}_1$, on the other hand, changes sign at $\bar{t} = 0$, and it has extrema at

$$\bar{t}_\pm = \pm b/\sqrt{2}v\gamma(v) \tag{4.113}$$

(so for large velocities this is a narrow time interval) with

$$\bar{E}_1(\bar{x}^i_P, \bar{t}_\pm) = \pm\frac{2Q}{3\sqrt{3}b^2} \tag{4.114}$$

(which is independent of $\vec{v}$).

For the magnetic field, one sees that $\bar{B}_1 = \bar{B}_2 = 0$, but that there is a non-zero component

$$\bar{B}_3 = -\beta\gamma E_2/c = -\beta\bar{E}_2/c \tag{4.115}$$

of the magnetic field in the x^3-direction orthogonal to both the electric field and the velocity of the charge. This reflects what is known as the Biot-Savart law of magnetostatics. For an arbitrary direction of the velocity $\vec{v}$ the result can be written as

$$\overline{\vec{B}} = (\vec{v} \times \overline{\vec{E}})/c^2 \ . \tag{4.116}$$

In a similar way one can determine the electromagnetic field produced by a steady (constant velocity) current from the simple electrostatic field of a charged wire. In particular, this means that the magnetic field generated by a current can be regarded as a relativistic effect. Even though the typical velocities in a current, of the order $v \sim \mathcal{O}(1mm/s) \ll c$, are very far from what one would usually call "relativistic velocities", this is a very visible and common effect (electric motors!), because of the large (Avogadro-ish) number of charge carriers in a current which all contribute to the magnetic field.

4.11 Covariant Formulation of the Lorentz Force Equation

The non-relativistic (better: Galilean relativistic) equation of motion for a massive charged particle with mass m and charge q in an electromagnetic field is

$$\frac{d}{dt}(m\vec{v}) = q(\vec{E} + \vec{v} \times \vec{B}) \ , \tag{4.117}$$

where the force term on the right-hand side is known as the Lorentz force. Taking the scalar product of this equation with $\vec{v}$, one finds

$$\frac{d}{dt}(m\vec{v}^2/2) = q\vec{E}.\vec{v} \ , \tag{4.118}$$

which describes the change in the kinetic energy of the particle due to the work done on it by the electric field.

We already know how to modify the left-hand side of (4.117) in order to obtain a Lorentz-tensorial expression: we replace the velocity $\vec{v}$ by the 4-velocity u^a and the derivative with respect to time by the derivative with respect to proper time,

$$\frac{d}{dt}(m\vec{v}) \to \frac{d}{dt}(m\gamma(v)\vec{v}) = \frac{d}{dt}\vec{p} \to \frac{d}{d\tau}p^a = \frac{d}{d\tau}(mu^a) \ . \tag{4.119}$$

What about the right-hand side? In order to reproduce this we evidently need to construct a 4-vector that is linear in F_{ab} and linear in u^a. There are not so many possiblilities for this. In fact, up to signs and factors the only possibility is $F^{ab}u_b$. Let us calculate the spatial components of this:

$$F^{ib}u_b = F^{i0}u_0 + F^{ij}u_j = (-E_i/c)(-\gamma(v)c) + \epsilon_{ijk}\gamma(v)v^jB^k = \gamma(v)(\vec{E} + \vec{v} \times \vec{B})_i \ . \tag{4.120}$$

We see that, up to the γ-factor, we find on the nose and very naturally the rather peculiar Lorentz force term. We can thus write down our candidate Lorentz invariant equation of motion for a charged particle in the Maxwell field, namely

$$\frac{d}{d\tau}p^a = qF^{ab}u_b \ . \tag{4.121}$$

In section 4.12 below, we will derive (4.121) from a Lorentz- and gauge invariant action principle for a charged particle coupled to the Maxwell field.

Remarks:

1. Using the fact that $\gamma(v)$ is the conversion factor between $d\tau$ and dt, we see that the spatial components of this equation can be written as

$$\gamma(v)\frac{d}{dt}\vec{p} = \gamma(v)q(\vec{E} + \vec{v} \times \vec{B}) \quad \Leftrightarrow \quad \frac{d}{dt}\vec{p} = q(\vec{E} + \vec{v} \times \vec{B}) \tag{4.122}$$

This differs from the non-relativistic equation (4.117) only by the replacement $m\vec{v} \to \vec{p} = m\gamma(v)\vec{v}$ on the left-hand side, while the right-hand Maxwell sides of the two equations are identical. In particular, this equation has the correct non-relativistic limit.

2. We noted before, in section 3.3, that any candidate equation of the form

$$\frac{d}{d\tau}p^a = K^a \tag{4.123}$$

requires the force to be orthogonal to the 4-velocity,

$$\frac{d}{d\tau}p^a = ma^a = K^a \quad \Rightarrow \quad K^a u_a = 0 \ . \tag{4.124}$$

In the case at hand, this is indeed satisfied,

$$K^a = qF^{ab}u_b \quad \Rightarrow \quad K^a u_a = qF^{ab}u_a u_b = 0 \tag{4.125}$$

by anti-symmetry of F^{ab} and symmetry of $u_a u_b$.

3. It remains to discuss the temporal component of (4.121). It can be written as

$$\frac{d}{dt}p^0 = qF^{0k}u_k/\gamma(v) = q\vec{E}.\vec{v}/c \quad \Leftrightarrow \quad \frac{d}{dt}E = q\vec{E}.\vec{v} \tag{4.126}$$

where $E = m\gamma(v)c^2$, and can therefore, exactly as (4.118), be interpreted as the change in the energy E of the particle due to the work performed on the particle by the electric field.

4. Just as (4.118) was implied by (4.117), in the present case (and in general for any K^a), one has

$$\frac{d}{d\tau}p^i = K^i \quad \Rightarrow \quad \frac{d}{d\tau}p^0 = K^0 \ . \tag{4.127}$$

This is best understood as a consequence of the fact that the 4 components of K^a are not independent,

$$K^a u_a = 0 \quad \Leftrightarrow \quad K^0 = -K^i u_i/u_0 \ . \tag{4.128}$$

Indeed, using the spatial components of the equation of motion, one finds an equation which is independent of the K^a,

$$\frac{d}{d\tau}p^0 = -K^i u_i/u_0 = -(\frac{d}{d\tau}p^i)/u_0 \quad \Leftrightarrow \quad u_a\frac{d}{d\tau}p^a = 0 \ , \tag{4.129}$$

and which is of course just the identity that 4-velocity and 4-acceleration are orthogonal, $u_a a^a = 0$.

We now want to look at the Lorentz force equation from the point of view of an action principle. This is rather straightforward, and it is also very instructive as it teaches us how to introduce forces / interactions in a free (non-interacting) matter theory in a Lorentz invariant manner by coupling the matter (here particles) to gauge fields in a Lorentz and gauge invariant way.

As a reminder, the action for a free relativistic particle was (we now use the subscript 0 on S_0 to indicate that this is the free action)

$$S_0[x] = -mc^2 \int d\tau \ . \tag{4.130}$$

with

$$\delta S_0[x] = \int d\tau \left(-\frac{d}{d\tau} p_a \right) \delta x^a \quad \Rightarrow \quad \frac{d}{d\tau} p_a = 0 \ . \tag{4.131}$$

We also know from the previous section that the equation of motion for a charged particle in the Maxwell field is

$$\frac{d}{d\tau} p_a = q F_{ab} u^b = q F_{ab} \dot{x}^b \ . \tag{4.132}$$

It is evident that in order to derive this equation from an action principle, we need to couple the particle to the Maxwell field. The action will thus take the form

$$S[x; A] = S_0[x] + S_I[x; A] \ , \tag{4.133}$$

where the 2nd term $S_I[x; A]$ describes the coupling (interaction) between particle and field, and I use the notation $S[x; A]$ to indicate that the action should depend on the gauge field $A_a(x)$, but that A_a is not, at this point, a dynamical variable that is to be varied separately. So our aim is to determine $S_I[x; A]$.

The low-brow (and perhaps not very insightful) way to go about this is to remind oneself how this is done in the non-relativistic case, and to then continue from there. Thus the coupling to an electric field is simply described by adding to the Lagrangian minus the potential electrostatic energy, which is nothing other than

$$V = q\phi \tag{4.134}$$

with ϕ the eletric potential (it is no coincidence that potentials are called potentials!). To describe the coupling to the magnetic field, one needs to introduce a (from the point of view of classical non-relativistic mechanics) rather peculiar velocity-dependent potential as well,

$$V = q\phi - q\vec{A}.\vec{v} \ . \tag{4.135}$$

Then one can show that the Euler-Lagrange equations resulting from

$$S = \int dt \ \left(\frac{m}{2} \vec{v}^2 - V \right) \tag{4.136}$$

are indeed precisely the Lorentz force equations (4.117).

One could then observe that, with our definition of A_a, the 2 terms in V can be combined into

$$-V = q(A_0 c + A_i v^i) = q A_a \frac{dx^a}{dt} \ , \tag{4.137}$$

and one might then perhaps be led to guess that the correct relativistic interaction action is

$$S_I[x;A] \stackrel{?}{=} q \int d\tau \; A_a \dot{x}^a \quad . \tag{4.138}$$

While this guess turns out to be correct, it is much more instructive to think about this (and arrive at this result) in a very different way, which requires no prior non-relativistic knowledge.

Our building blocks are $x^a = x^a(\tau), \dot{x}^a$ etc. for the particle, and A_a, F_{ab} etc. for the Maxwell field, and our aim is to find the simplest action that gives rise to Lorentz and gauge invariant equations of motion (and "simplest" here means lowest number of derivatives, lowest degree polynomial etc.).

Perhaps the simplest candidate for the interaction Lagrangian is $A_a x^a$. This is evidently Lorentz invariant, but equally evidently it will give rise to a contribution $\sim A_a$ to the force, which is not gauge invariant, and hence we discard it.

The next simplest term is $A_a \dot{x}^a$. This is again evidently Lorentz invariant, but what about gauge invariance? Under a gauge transformation $A_a \to A_a + \partial_a \Psi$ we find

$$A_a \dot{x}^a \to A_a \dot{x}^a + (\partial_a \Psi)\dot{x}^a = A_a \dot{x}^a + \frac{d}{d\tau}\Psi \quad . \tag{4.139}$$

Thus, even though $A_a \dot{x}^a$ is not gauge invariant, very cooperatively $A_a \dot{x}^a$ is gauge invariant up to a total derivative. Therefore the action only changes by a boundary term, and since this has no impact on the equations of motion, this is sufficient to ensure gauge invariance of the equations ot motion.

Therefore we postulate the action

$$S_I[x;A] = q \int d\tau \; A_a \dot{x}^a \quad . \tag{4.140}$$

We see that this agrees with the guess (4.138).

It is now straightforward to derive that the Euler-Lagrange equations derived from the action $S_0[x] + S_I[x;A]$ are indeed precisely the relativistic Lorentz force equations (4.132). Let us do this first, and then I will add some more comments on this action.

Since we already know the variation of $S_0[x]$, we just need to determine that of $S_I[x;A]$. For that we use that the variation of the 4-velocity is

$$\delta \dot{x}^a = \frac{d}{d\tau}\delta x^a \quad , \tag{4.141}$$

and that the variation of $A_a(x)$ induced by a variation $x^a \to x^a + \delta x^a$ is

$$\delta A_a = (\partial_b A_a)\delta x^b \quad . \tag{4.142}$$

We will also use

$$\frac{d}{d\tau} A_a = (\partial_b A_a)\dot{x}^b \quad . \tag{4.143}$$

With this we can calculate (using integration by parts and, as usual, dropping the boundary term)

$$\begin{aligned}
\delta \int d\tau\ A_a \dot{x}^a &= \int d\tau \left((\partial_b A_a) \delta x^b \dot{x}^a + A_a \delta \dot{x}^a \right) \\
&= \int d\tau \left((\partial_a A_b) \delta x^a \dot{x}^b - \delta x^a \frac{d}{d\tau} A_a \right) \\
&= \int d\tau \left((\partial_a A_b - \partial_b A_a) \delta x^a \dot{x}^b \right) \\
&= \int d\tau\ F_{ab} \delta x^a \dot{x}^b \quad .
\end{aligned} \tag{4.144}$$

Thus combining this with (4.131) we find

$$\delta(S_0[x] + S_I[x, A]) = \int d\tau \left(-\frac{d}{d\tau} p_a + q F_{ab} \dot{x}^b \right) \delta x^a \tag{4.145}$$

and therefore the Euler-Lagrange equations are precisely the Lorentz force equations (4.132).

<u>Remarks:</u>

1. The rationale for introducing the charge q in front of the action (4.140) is that it is the *coupling constant*, i.e. a measure of the strength of the interaction between the particle and the Maxwell field (in particular, for an uncharged particle, $q = 0$, there is no such interaction).

2. Note that the momenta p_a in the above discussion are the covariant conjugate momenta of the free particle, i.e. $p_a = m u_a$. Because of the velocity dependendence of the interaction Lagrangian, these are not the same as the covariant conjugate momenta P_a associated to the sum of the free and interaction Lagrangian,

$$L = L_0 + L_I \quad \Rightarrow \quad P_a = \frac{\partial L}{\partial \dot{x}^a} = p_a + q A_a \quad . \tag{4.146}$$

   The modification of the spatial components is already familiar from non-relativistic mechanics. Thus the quantity of interest is the temporal component

$$P^0 = p^0 + q A^0 = (E + q\phi)/c = (m\gamma(v)c^2 + q\phi)/c \quad . \tag{4.147}$$

   This is the total (relativistic kinetic plus electric potential) energy of the particle.

3. The interaction action can be written as just the line integral of $A = A_a dx^a$ over the worldline (curve) C of the particle,

$$S_I[x; A] = q \int d\tau\ A_a \dot{x}^a = q \int_C A_a dx^a \equiv q \int_C A \quad . \tag{4.148}$$

   Since one can integrate $A = A_a dx^a$ in a natural way only over 1-dimensional spaces, this makes it clear that the elementary objects that carry electric charge and that A_a can couple to are objects with 1-dimensional worldlines, i.e. particles. For some comments on generalisations of this kind of reasoning to other, more exotic, situations see section 7.1.

4. At this point it is natural to wonder if one can derive not just the Lorentz force equation but also the Maxwell equations themselves from an action principle. This is (of course) indeed the case, but requires an extension of action principles and variational calculus to field theories. This will be the subject of section 5.