Solutions to Assignments 01

1. The Lorentz Group

   (a) The first claim follows from multiplicativity of the determinant (and invariance under transposition):

   $$L^T\eta L=\eta \quad\Rightarrow\quad \det(L^T\eta L)=\det(\eta) \quad\Rightarrow\quad \det(L)^2=+1\ . \tag{1}$$

   The second claim follows fom writing $(L^T\eta L)_{00}=\eta_{00}$ explicitly,

   $$\begin{aligned} &\eta_{\alpha\beta}L^\alpha_0 L^\beta_{\ 0} \stackrel{!}{=} \eta_{00}=-1\\ \Rightarrow\quad &\eta_{00}L^0_0L^0_{\ 0}+\eta_{ik}L^i_0L^k_{\ 0}=-(L^0_0)^2+\delta_{ik}L^i_0L^k_{\ 0}=-1\\ \Rightarrow\quad &(L^0_0)^2=1+\delta_{ik}L^i_0L^k_{\ 0}\geq 1\ . \end{aligned} \tag{2}$$

   (b) It is trivial to verify that

   $$L_1^T\eta L_1=\eta, L_2^T\eta L_2=\eta \quad\Rightarrow\quad (L_1L_2)^T\eta(L_1L_2)=\eta\ . \tag{3}$$

   Existence of an inverse L^{-1} follows from $\det L\neq 0$ (shown above). That $L\in\mathcal{L}\Rightarrow L^{-1}\in\mathcal{L}$ follows from

   $$L^T\eta L=\eta \quad\Leftrightarrow\quad \eta=(L^{-1})^T\eta L^{-1}\ . \tag{4}$$

2. Tensor Algebra: Lorentz Tensors

   By definition a Lorentz vector transforms as

   $$\bar{v}^\alpha=L^\alpha_{\ \beta}v^\beta\ , \tag{5}$$

   and a Lorentz covector as

   $$\bar{u}_\alpha=\Lambda_\alpha^{\ \beta}u_\beta\ , \tag{6}$$

   with

   $$\Lambda=(L^T)^{-1} \quad\Leftrightarrow\quad \Lambda_\alpha^{\ \beta}L^\alpha_{\ \gamma}=\delta^\beta_{\ \gamma}\ . \tag{7}$$

   This definition is such that the *contraction* between a vector and a covector is a *scalar* (invariant under Lorentz transformations),

   $$\bar{u}_\alpha\bar{v}^\alpha=\Lambda_\alpha^{\ \beta}L^\alpha_{\ \gamma}u_\beta v^\gamma=\delta^\beta_{\ \gamma}u_\beta v^\gamma=u_\beta v^\beta=u_\alpha v^\alpha\ . \tag{8}$$

   Higher rank tensors transform like products of vectors and covectors, i.e. a (p,q) tensor transforms with p factors of L and q factors of Λ and is written as an object with p upper indices and q lower indices.

   By the same calculation as above one then finds that any contracted pair of indices on a tensor (summation over one "upper" and one "lower" index) is invariant. Therefore the tensor type of the resulting object can be read off just by looking at the number of uncontracted upper and lower indices. For example:

(a) for the contraction of a $(2,0)$-tensor and a $(0,1)$-tensor (covector) one has

$$\bar{T}^{\alpha\beta}\bar{u}_\beta = L^\alpha_{\ \gamma}L^\beta_{\ \delta}\, T^{\gamma\delta}\, \Lambda_\beta^{\ \rho}u_\rho = L^\alpha_{\ \gamma}\,\delta^\rho_{\ \delta}\, T^{\gamma\delta}u_\rho = L^\alpha_{\ \gamma}\left(T^{\gamma\delta}u_\delta\right) \tag{9}$$

so that $T^{\alpha\beta}u_\beta$ transforms like (and therefore is) a $(1,0)$-tensor (vector), as indicated by the fact that this object has one free upper index.

(b) likewise the trace of a $(1,1)$-tensor is a scalar,

$$\bar{T}^\alpha_{\ \alpha} = L^\alpha_{\ \beta}\Lambda_\alpha^{\ \gamma}T^\beta_{\ \gamma} = \delta^\gamma_{\ \beta}\, T^\beta_{\ \gamma} = T^\beta_{\ \beta} \tag{10}$$

Note that the trace of a $(0,2)$-tensor $T_{\alpha\beta}$ is not well-defined without using the Minkowski metric, i.e. something like

$$\mathsf{trace}(T_{\alpha\beta}) \stackrel{?}{=} \sum_\alpha T_{\alpha\alpha} \qquad (???) \tag{11}$$

is not Lorentz-invariant and therefore depends on the inertial system in which it is evaluated. However, with the help of the Minkowski metric one can define a Lorentz-invariant trace (i.e. a scalar) via

$$T_{\alpha\beta} \to T^\alpha_{\ \beta} = \eta^{\alpha\gamma}T_{\gamma\beta} \to T^\alpha_{\ \alpha} = \eta^{\alpha\gamma}T_{\gamma\alpha} \qquad \checkmark \tag{12}$$

("taking the trace with respect to η"). This is now manifestly a scalar.

3. Tensor Analysis: Lorentz Tensors and their Derivatives

As recalled in the previous exercise, the formalism is designed in such a way that the transformation behaviour (tensorial nature) can just be read off from the free indices. This extends to partial derivatives of tensors.

(a) In particular, the partial derivative $(\partial/\partial x^\alpha) = \partial_\alpha$ transforms as a covector,

$$\bar{\partial}_\alpha \equiv \frac{\partial}{\partial \bar{x}^\alpha} = \Lambda_\alpha^{\ \beta}\partial_\beta \ . \tag{13}$$

Proof:

$$\begin{aligned} \bar{x}^\alpha = L^\alpha_{\ \beta}x^\beta \quad &\Rightarrow \quad \frac{\partial}{\partial x^\beta} = \frac{\partial \bar{x}^\alpha}{\partial x^\beta}\frac{\partial}{\partial \bar{x}^\alpha} = L^\alpha_{\ \beta}\frac{\partial}{\partial \bar{x}^\alpha} = (L^T)_\beta^{\ \alpha}\frac{\partial}{\partial \bar{x}^\alpha} \\ &\Rightarrow \quad \frac{\partial}{\partial \bar{x}^\alpha} = ((L^T)^{-1})_\alpha^{\ \beta}\frac{\partial}{\partial x^\beta} = \Lambda_\alpha^{\ \beta}\frac{\partial}{\partial x^\beta} \ . \end{aligned} \tag{14}$$

It follows that the partial derivative of a scalar function f, i.e. $\bar{f}(\bar{x}) = f(x)$, is a covector.

(b) More generally, ∂_α acting on a (p,q)-tensor gives a $(p,q+1)$-tensor. Then the answers are immediately $V^\alpha\partial_\alpha f$ scalar, $V^\alpha\partial_\beta f$ (1,1)-tensor, $\partial_\alpha V^\alpha$ scalar (the covariant divergence), $f\partial_\alpha V^\alpha$ scalar, $\partial_\alpha V_\beta$ (0,2)-tensor, $\partial_\alpha\partial_\beta f$ (0,2)-tensor, and $V^\alpha\partial_\beta V_\alpha$ covector.