KFT Solutions 02

1. Action for a free particle

   The action is

$$S[x] = -mc^2 \int d\tau = -mc^2 \int d\lambda (d\tau/d\lambda) \equiv \int d\lambda\, L_\lambda \tag{1}$$

   with $L_\lambda = -mc^2(d\tau/d\lambda)$.

   (a) The momentum is

$$\begin{aligned} p_\alpha = \frac{\partial L_\lambda}{\partial x'^\alpha} &= -mc\, \tfrac{1}{2} \left(-\eta_{\alpha\beta} x'^\alpha x'^\beta\right)^{-1/2} \left(-2\eta_{\alpha\beta} x'^\beta\right) \\ &= mc\, \eta_{\alpha\beta} \left(\frac{c d\tau}{d\lambda}\right)^{-1} \frac{dx^\beta}{d\lambda} = m\eta_{\alpha\beta} \frac{dx^\beta}{d\tau} = mu_\alpha \end{aligned} \tag{2}$$

   or $p^\alpha = mu^\alpha$. In an inertial system with cordinates $(x^0 = ct, x^k)$, and with $v^k = dx^k/dt$ one has

$$p^0 = m\gamma(v)c = E/c \quad , \quad p^k = m\gamma(v)v^k \quad . \tag{3}$$

   (b) The Euler-Lagrange equation is

$$\frac{d}{d\lambda}\frac{\partial L_\lambda}{\partial x'^\alpha} - \frac{\partial L_\lambda}{\partial x^\alpha} = \frac{d}{d\lambda}\frac{\partial L_\lambda}{\partial x'^\alpha} = 0 \tag{4}$$

   and

$$\frac{d}{d\lambda}\frac{\partial L_\lambda}{\partial x'^\alpha} = \frac{d\tau}{d\lambda}\frac{d}{d\tau} mu_\alpha = \left(m\eta_{\alpha\beta}\frac{d\tau}{d\lambda}\right)\frac{d^2x^\beta}{d\tau^2} = 0 \quad \Leftrightarrow \quad \frac{d^2x^\beta}{d\tau^2} = 0 \quad . \tag{5}$$

   (c) The Lagrangian is

$$L_t = -mc^2\frac{d\tau}{dt} = -mc^2\sqrt{1 - \vec{v}^2/c^2} = -mc^2\gamma(v)^{-1} \quad . \tag{6}$$

   Thus the canonical momenta are

$$p_k^{(c)} = \frac{\partial L_t}{\partial v^k} = (-mc^2)\gamma(v)(-v^k/c^2) = m\gamma(v)v^k = p^k \quad , \tag{7}$$

   and the canonical Hamiltonian is

$$\begin{aligned} H &= p_k^{(c)} v^k - L_t = m\gamma(v)\vec{v}^2 + mc^2\gamma(v)^{-1} \\ &= m\gamma(v)(\vec{v}^2 + c^2(1 - \vec{v}^2/c^2)) = m\gamma(v)c^2 = E = cp^0 \quad . \end{aligned} \tag{8}$$

   (d) The covariant Hamiltonian is

$$\mathcal{H}_\lambda = p_\alpha \frac{dx^\alpha}{d\lambda} - L_\lambda = p_\alpha \frac{dx^\alpha}{d\tau}\frac{d\tau}{d\lambda} + mc^2\frac{d\tau}{d\lambda} \quad . \tag{9}$$

   With $p^\alpha = mdx^\alpha/d\tau$ this can be written as

$$\mathcal{H}_\lambda = \frac{1}{m}\left(p^\alpha p_\alpha + m^2c^2\right)\frac{d\tau}{d\lambda} \tag{10}$$

   which proves both assertions, namely that $\mathcal{H}_\lambda = 0$ and that this is equivalent to the mass-shell condition.