KFT Solutions 03

1. Inhomogeneous Maxwell-Equations and Potentials

(a) Under the gauge transformation $A_\beta \to A_\beta + \partial_\beta \Psi$, $F_{\alpha\beta}$ transforms as

$$\partial_\alpha A_\beta - \partial_\beta A_\alpha \to \partial_\alpha A_\beta + \partial_\alpha \partial_\beta \Psi - \partial_\beta A_\alpha - \partial_\beta \partial_\alpha \Psi = \partial_\alpha A_\beta - \partial_\beta A_\alpha \tag{1}$$

and therefore $F_{\alpha\beta}$ is gauge-invariant.

(b) With $A_\alpha = (-\phi/c, \vec{A})$ one has

$$\begin{aligned} F_{0k} &= -F_{k0} = \partial_0 A_k - \partial_k A_0 = c^{-1}(\partial_t A_k + \partial_k \phi) = -E_k/c \\ F_{ik} &= \partial_i A_k - \partial_k A_i = \epsilon_{ik\ell} B_\ell \quad (F_{12} = B_3 \quad \text{etc.}) \end{aligned} \tag{2}$$

and therefore, with $F^{\alpha\beta} = \eta^{\alpha\gamma}\eta^{\beta\delta}F_{\gamma\delta}$,

$$F^{0k} = -F^{k0} = -F_{0k} = E_k/c \quad , \quad F^{ik} = F_{ik} = \epsilon_{ik\ell} B_\ell \;\; . \tag{3}$$

(c) Thus, with $J^\alpha = (\rho c, \vec{J})$ one has

$$\partial_\alpha F^{\alpha 0} = \partial_k F^{k0} = -c^{-1}\vec{\nabla}.\vec{E} = -\rho/(\epsilon_0 c) = -\mu_0 c \rho = -\mu_0 J^0 \tag{4}$$

and

$$\begin{aligned} \partial_\alpha F^{\alpha 1} &= \partial_0 F^{01} + \partial_2 F^{21} + \partial_3 F^{31} = c^{-2}\partial_t E_1 - \partial_2 B_3 + \partial_3 B_2 \\ &= -(\vec{\nabla} \times \vec{B} - \frac{1}{c^2}\partial_t \vec{E})_1 = -\mu_0 J_1 = -\mu_0 J^1 \end{aligned} \tag{5}$$

(and likewise for the 2- and 3-components).

(d) One has

$$\partial_\alpha F^{\alpha\beta} = \partial_\alpha \partial^\alpha A^\beta - \partial_\alpha \partial^\beta A^\alpha = \Box A^\beta - \partial^\beta \partial_\alpha A^\alpha = -\mu_0 J^\beta \tag{6}$$

and therefore $\Box A_\beta - \partial_\beta \partial_\alpha A^\alpha = -\mu_0 J_\beta$.

(e) One has

$$\Box(A_\beta + \partial_\beta \Psi) - \partial_\beta \partial_\alpha (A^\alpha + \partial^\alpha \Psi) = \Box A_\beta + \partial_\beta \Box \Psi - \partial_\beta \partial_\alpha A^\alpha - \partial_\beta \Box \Psi \tag{7}$$

Since the $\Box\Psi$-term cancels, the expression is gauge invariant.

(f) From $\partial_\alpha F^{\alpha\beta} = -\mu_0 J^\beta$ one deduces $-\mu_0 \partial_\beta J^\beta = \partial_\beta \partial_\alpha F^{\alpha\beta} = 0$ because $F^{\alpha\beta} = -F^{\beta\alpha}$ is anti-symmetric while $\partial_\alpha \partial_\beta = \partial_\beta \partial_\alpha$ is symmetric.

2. THE HOMOGENENEOUS MAXWELL-EQUATIONS

   (a) One has $\partial_\alpha F_{\beta\gamma} = \partial_\alpha \partial_\beta A_\gamma - \partial_\alpha \partial_\gamma A_\beta$ etc. Using the fact that 2nd partial derivatives commute one deduces

$$\begin{aligned} & \partial_\alpha F_{\beta\gamma} + \partial_\gamma F_{\alpha\beta} + \partial_\beta F_{\gamma\alpha} \\ = \; & \partial_\alpha \partial_\beta A_\gamma - \partial_\gamma \partial_\alpha A_\beta + \partial_\gamma \partial_\alpha A_\beta - \partial_\beta \partial_\gamma A_\alpha + \partial_\beta \partial_\gamma A_\alpha - \partial_\alpha \partial_\beta A_\gamma = 0 \end{aligned} \tag{8}$$

   (b) $\partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} + \partial_\gamma F_{\alpha\beta} = 0$

      i. Two indices equal ($\alpha = \beta$, say): $\partial_\alpha F_{\alpha\gamma} + \partial_\alpha F_{\gamma\alpha} + \partial_\gamma F_{\alpha\alpha} = 0$ is identically satisfied because $F_{\alpha\alpha} = 0$, $F_{\alpha\gamma} + F_{\gamma\alpha} = 0$.

      ii. All 3 indices spatial, $(\alpha, \beta, \gamma) = (1, 2, 3)$:

$$\partial_1 F_{23} + \partial_2 F_{31} + \partial_3 F_{21} = \partial_1 B_1 + \partial_2 B_2 + \partial_3 B_3 = \vec{\nabla}.\vec{B} = 0 \tag{9}$$

      iii. One index time, the others spatial, e.g. $(\alpha, \beta, \gamma) = (0, 1, 2)$:

$$\partial_0 F_{12} + \partial_1 F_{20} + \partial_2 F_{01} = c^{-1}(\partial_t B_3 + \partial_1 E_2 - \partial_2 E_1) = c^{-1}(\vec{\nabla} \times \vec{E} + \partial_t \vec{B})_3 = 0 \tag{10}$$

      (and likewise for the other components).

3. THE DUAL FIELD STRENGTH TENSOR

   The dual field strength tensor is defined by

$$\tilde{F}^{\alpha\beta} = \tfrac{1}{2}\epsilon^{\alpha\beta\gamma\delta} F_{\gamma\delta} \quad . \tag{11}$$

   With $F_{0i} = -c^{-1}E_i$, $F_{ij} = \epsilon_{ijk}B_k$ and $\tilde{F}^{\alpha\beta} = \frac{1}{2}\epsilon^{\alpha\beta\gamma\delta}F_{\gamma\delta}$ one gets for the non-zero components of $\tilde{F}^{\alpha\beta}$:

$$\tilde{F}^{0i} = \tfrac{1}{2}\epsilon^{0i\gamma\delta} F_{\gamma\delta} = \tfrac{1}{2}\epsilon^{0ijk} F_{jk} = \tfrac{1}{2}\epsilon^{0ijk}\epsilon_{jkl} B_l = -B_i \tag{12}$$

$$\tilde{F}^{ij} = \tfrac{1}{2}\epsilon^{ij\gamma\delta} F_{\gamma\delta} = \epsilon^{ij0k} F_{0k} = -c^{-1}\epsilon^{ij0k} E_k = c^{-1}\epsilon^{ijk} E_k \; . \tag{13}$$

   The equation $\partial_\lambda \tilde{F}^{\lambda\delta} = 0$ can then be written as

$$\partial_\lambda \tilde{F}^{\lambda 0} = -\partial_i \tilde{F}^{0i} = \vec{\nabla} \cdot \vec{B} = 0 \tag{14}$$

$$\begin{aligned} \partial_\lambda \tilde{F}^{\lambda j} &= \partial_0 \tilde{F}^{0j} + \partial_i \tilde{F}^{ij} = -c^{-1}\partial_t B_j - c^{-1}\epsilon_{jik}\partial_i E_k \\ &= -\frac{1}{c}\left(\partial_t \vec{B} + \vec{\nabla} \times \vec{E}\right)_j = 0 \quad , \end{aligned} \tag{15}$$

   which proves the assertion.