KFT Solutions 04

1. Lorentz Invariants

   (a)

$$
\begin{aligned}
I_1 &= \tfrac{1}{4}F_{\alpha\beta}F^{\alpha\beta} = \tfrac{1}{4}\left(F_{0i}F^{0i} + F_{i0}F^{i0} + F_{ij}F^{ij}\right) \\
&= \tfrac{1}{2}\left(-(F_{0i})^2 + (F_{ij})^2\right) = \tfrac{1}{2}\left(\vec{B}^2 - c^{-2}\vec{E}^2\right) \qquad (1) \\
I_2 &= \tfrac{1}{4}F_{\alpha\beta}\tilde{F}^{\alpha\beta} = \tfrac{1}{4}\left(F_{0i}\tilde{F}^{0i} + F_{i0}\tilde{F}^{i0} + F_{ij}\tilde{F}^{ij}\right) \\
&= \tfrac{1}{4}\left(2c^{-1}E_iB_i + \epsilon_{ijk}B_kc^{-1}\epsilon^{ijl}E_l\right) = c^{-1}\vec{E}\cdot\vec{B} \qquad (2)
\end{aligned}
$$

   where one has used $\epsilon^{ijl}\epsilon_{ijk} = 2\delta^l_k$. If $\vec{E} = 0$ in one inertial system, then $I_1 > 0$ and $I_2 = 0$ in all inertial systems, and thus $\vec{E}.\vec{B} = 0$ and $|\vec{E}| < |\vec{B}|$ in all inertial systems.

   (b)

$$
8I_2 = \epsilon^{\alpha\beta\gamma\delta}F_{\alpha\beta}F_{\gamma\delta} = 2\epsilon^{\alpha\beta\gamma\delta}(\partial_\alpha A_\beta)F_{\gamma\delta} = 2\partial_\alpha\left(\epsilon^{\alpha\beta\gamma\delta}A_\beta F_{\gamma\delta}\right) \qquad (3)
$$

   because $\epsilon^{\alpha\beta\gamma\delta}\partial_\alpha F_{\gamma\delta} = 0$ (Bianchi identity). Thus I_2 is a total derivative, $I_2 = \partial_\alpha C^\alpha$. C^α is not gauge invariant but changes by a total derivative under a gauge transformation,

$$
\epsilon^{\alpha\beta\gamma\delta}A_\beta F_{\gamma\delta} \to \epsilon^{\alpha\beta\gamma\delta}\partial_\beta\psi F_{\gamma\delta} = \partial_\beta(\epsilon^{\alpha\beta\gamma\delta}\psi F_{\gamma\delta}) \qquad (4)
$$

2. Lorentz transformation of $\vec{B}$:

   Using $\bar{F}_{\alpha\beta} = \Lambda_\alpha^\gamma\Lambda_\beta^\delta F_{\gamma\delta}$ one wants to compute the transformation of $\bar{F}_{ij}$ which contains the magnetic field components. To do this we need $\Lambda_\alpha{}^\beta$ which is obtained from L^α_β by inverse transposition, which gives

$$
\left(\Lambda_\alpha{}^\beta\right) = \begin{pmatrix} \cosh\alpha & \sinh\alpha & 0 & 0 \\ \sinh\alpha & \cosh\alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} . \qquad (5)
$$

   With it one computes :

$$
\begin{aligned}
\bar{F}_{ij} &= \Lambda_i{}^\gamma\Lambda_j{}^\delta F_{\gamma\delta} = \Lambda_i{}^0\Lambda_j{}^lF_{0l} + \Lambda_i{}^k\Lambda_j{}^0F_{k0} + \Lambda_i{}^k\Lambda_j{}^lF_{kl} \\
&= \left(\Lambda_i{}^0\Lambda_j{}^l - \Lambda_i{}^l\Lambda_j{}^0\right)F_{0l} + \Lambda_i{}^k\Lambda_j{}^lF_{kl} \qquad (6)
\end{aligned}
$$

   such that

$$
\begin{aligned}
\bar{F}_{12} &= \sinh\alpha F_{02} + \cosh\alpha F_{12} = -c^{-1}\gamma\beta E_2 + \gamma B_3 \qquad (7) \\
\bar{F}_{23} &= F_{23} \qquad (8) \\
\bar{F}_{31} &= -\sinh\alpha F_{03} + \cosh\alpha F_{31} = c^{-1}\gamma\beta E_3 + \gamma B_2 \qquad (9)
\end{aligned}
$$

from which one can read off the transformation of the magnetic field :

$$\begin{aligned}
\bar{B}_1 &= B_1 \\
\bar{B}_2 &= \gamma B_2 + c^{-1}\beta\gamma E_3 \\
\bar{B}_3 &= \gamma B_3 - c^{-1}\beta\gamma E_2
\end{aligned} \qquad (10)$$